
An Optimal Control Problem Governed by a Regularized Phase-Field Fracture Propagation Model

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AN OPTIMAL CONTROL PROBLEM GOVERNED BY A REGULARIZED PHASE-FIELD FRACTURE PROPAGATION MODEL

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Abstract. This paper is concerned with an optimal control problem governed by a regularized fracture model using a phase-field technique. To avoid the non-differentiability due to the irreversibility constraint on the fracture growth, the phase-field fracture model is relaxed using a penalization approach. Existence of a solution to the penalized fracture model is shown and existence of at least one solution for the regularized optimal control problem is established. Moreover, the linearized fracture model is considered and used to establish first order necessary conditions as well as to discuss QP-approximations to the nonlinear optimization problem. A numerical example suggests that these can be used to obtain a fast convergent algorithm.

Key words. optimal control, regularized fracture model, phase-field, existence of solutions

AMS subject classifications. 49J21, 49K21, 74R10

1. Introduction. This paper presents an optimal control formulation for regularized fracture propagation problems using phase-field methods. To the best of the authors knowledge, optimization problems involving fracture propagation have been considered either purely numerically, see [27]. Alternatively, mathematical analysis has been considered in settings involving a fracture of fixed length, see [31], or a fracture with variable length but prescribed fracture path, see [36]. Due to the considered phase-field approach our model allows for arbitrary fracture-paths including changes in the fracture topology.

Presently, phase-field approaches for the simulation of fracture propagation are subject of intensive research in both mathematical theory and applications. Based on variational principles, they provide an elegant way to approximate lower-dimensional surfaces and discontinuities. Rewriting Griffith's model [23] for brittle fracture in terms of a variational formulation was first done in [18]. Later, these concepts have been complemented with numerical examples [13] and well-posedness results including fractures with linear [19] and nonlinear elasticity [37]. A summary has been compiled in [14]. In [41, 42], the authors refined modeling and material law assumptions to formulate an incremental thermodynamically consistent phase-field model for fracture propagation.

With regard to numerical analysis and computational methods important advances have been made first in [13], which was later supplemented with an analysis of the solution algorithm [12]; for a complete proof of that algorithm, we also refer to [15]. For a general Ambrosio-Tortorelli functional [3, 4], numerical analysis was done in a second paper by the same authors [16]. Recent results and new features of this solution algorithm have been presented in [38]. Parameter studies and a slight re-interpretation of the original model were performed in [34]. A solution approach using shape optimization has been presented in [1] and phase-field models for struc-

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tural optimization are discussed in [9]. Sophisticated examples and benchmarks from mechanical engineering, using the refined phase-field modeling, have been studied in multiple papers, see, e.g., [2, 10, 11, 25, 41, 42, 48]. Recent modeling and numerical studies adding non-homogeneous traction forces acting on the fracture surface were conducted in [45, 46, 50].

For numerical simulations using variational models of fracture, a general challenge is associated with the resolution of the (very small) phase-field parameter ε in relation to the spatial discretization parameter such that $h \ll \varepsilon$. Since uniform mesh refinement yields huge computational cost, local mesh adaptivity (possibly based on a posteriori error estimation proposed first in [15], and extended to anisotropic mesh adaptivity in [5]) is indispensable for fine resolution of the fracture. The extension to goal-oriented error estimation using dual-weighted residuals has been addressed in [51]. Another method that purely focuses on fine meshes in the crack region has been developed in [25] for 2D simulations and extended in [35] to 3D. All these studies show that local mesh refinement is a key ingredient for phase-field fracture, which is important for practical problems.

However, in the present paper, we focus on the coupling of a regularized fracture model with an additional outer optimization and analyze the well-posedness of this outer optimization problem. A task that has, to the best of the authors knowledge, not been considered previously. In more detail, the main difficulty in deriving necessary optimality conditions lies in the irreversibility of the fracture, giving a variational inequality as lower-level problem. To deal with the inherent non-differentiability, we will introduce an additional penalty approach for the fracture problem. This gives rise to a quasi-linear system as a side condition, a setting not often discussed in the literature. In particular, well-posedness of this relaxed irreversible fracture problem is not obvious since the phase-field is not immediately in L^∞ , in contrast to many other contributions dealing with the irreversibility, e.g., [12, 13, 34, 41].

The outline of this paper is as follows: In Section 2, we formulate the nonlinear forward problem for fracture propagation utilizing a phase-field ansatz, and introduce a regularization of the irreversibility condition for the growth by a penalty approach with parameter γ . The final model under consideration is then given in Section 2.5. Then we briefly state the outer optimization problem, in Section 3. Solvability of both the relaxed fracture propagation problem as well as the optimization problem is discussed in Section 4. In Section 5, we discuss the properties of the linearized relaxed phase-field model, and show that the linearization gives rise to a Fredholm operator. This observation is then used to derive first order necessary conditions for the relaxed nonlinear optimization problem, in Section 6, under a constraint qualification. In addition, in Section 7, we show that quadratic approximations to the nonlinear optimization problem are always well-posed and admit a unique solution that can be characterized by its first order necessary optimality conditions. Then, in Section 8, we present a numerical example indicating that indeed quadratic approximations give rise to a convergent algorithm.

2. The Nonlinear Phase-Field Fracture Problem and its Relaxation.

Following the model proposed in [13, 18, 41, 42], we consider a time discrete, but spatially continuous phase-field approach to model the growth of the fracture over time. For simplicity, in this paper, the fracture growth is controlled by traction forces acting on the boundary of the domain. This is motivated by prior work [46] where the forward propagation of the fracture was driven by such forces acting on the fracture boundary. The irreversibility of the fracture growth induces an obstacle like problem

in each time-point.

2.1. Notation. We consider a bounded domain $\Omega \subset \mathbb{R}^2$. Its boundary $\partial\Omega$ is decomposed into Γ_D and Γ_N satisfying

$$\mathcal{H}^{d-1}(\Gamma_D) \neq 0 \quad \text{and} \quad \mathcal{H}^{d-1}(\Gamma_N) \neq 0$$

where \mathcal{H}^{d-1} is the $d - 1$ -dimensional Hausdorff-measure. We introduce the space of admissible displacements $H_D^1(\Omega; \mathbb{R}^2) := \{v \in H^1(\Omega; \mathbb{R}^2) \mid v = 0 \text{ on } \Gamma_D\}$. We assume that $\Omega \cup \Gamma_N$ is regular in the sense of Gröger, cf. [24], compare [26, Remark 1.6] for a characterization in the case $\Omega \subset \mathbb{R}^2$ considered here. By (\cdot, \cdot) , we denote the usual L^2 scalar product and by $\|\cdot\|$ the corresponding norms.

Throughout the paper, c denotes a generic constant, which is independent of the relevant quantities, but may take a different value in each appearance, even in the same line. If we would like to emphasize the dependence of such a constant on a particular value, we do so by introducing an appropriate index, i.e., c_ε denotes a constant whose value depends on some parameter ε if the precise dependence is not relevant for the argument.

2.2. Brittle Fracture. Following Griffith's criterion for brittle fracture, we suppose that the fracture propagation occurs when the elastic energy restitution rate reaches its critical value G_c . If q is a force applied on Γ_N , assuming that the fracture \mathcal{C} is not reaching $\partial\Omega$, we define the following total energy

$$E(q; u, \mathcal{C}) = \frac{1}{2}(\mathbb{C}e(u), e(u))_{\Omega \setminus \mathcal{C}} - (q, u)_{\Gamma_N} + G_c \mathcal{H}^{d-1}(\mathcal{C}), \quad (2.1)$$

where u denotes the vector-valued displacement field, \mathbb{C} the elasticity tensor, and $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ the symmetric gradient. Furthermore, we restrict ourselves to the consideration of homogeneous Dirichlet data for the displacement u , for simplicity.

In the functional (2.1), the first term describes the bulk energy, the second term traction boundary (Neumann) forces, and the final term the surface fracture energy. The energy functional is then to be minimized with respect to the kinematically admissible displacements u and any fracture set satisfying the fracture growth condition. Furthermore, the crack path is subject to an irreversibility constraint in time; namely that the crack will not heal. The corresponding mathematical formulation of this constraint will follow in the next Section 2.3.

2.3. Time-Discrete Ambrosio-Tortorelli Regularization of the Fracture.

As it is common, we would like to avoid dealing with the set of admissible fractures \mathcal{C} . To regularize the Hausdorff-measure, we follow [3, 4] and introduce an auxiliary time-dependent variable (i.e., a phase-field for the fracture) φ , defined on $\Omega \times (0, T)$. Specifically, the fracture region is characterized by $\varphi = 0$ and the non-fractured zone by $\varphi = 1$. For $0 < \varphi < 1$, we deal with a transition zone, which has width ε on each side of the fracture path.

For given $1 > \varepsilon > 0$, the regularized fracture functional reads

$$\Gamma_\varepsilon(\varphi) = \frac{1}{2\varepsilon} \|1 - \varphi\|^2 + \frac{\varepsilon}{2} \|\nabla \varphi\|^2. \quad (2.2)$$

This regularization of $\mathcal{H}^{d-1}(\mathcal{C})$, in the sense of the Γ -limit when $\varepsilon \rightarrow 0$, was used in [13, 14].

Further, to define the displacement on Ω rather than on $\Omega \setminus \{\varphi = 0\}$ and to avoid degeneracy of the elastic energy, we need to introduce an additional regularization

parameter $\kappa > 0$ with $\kappa \ll \varepsilon$ essentially replacing the fracture by a softer material. With this parameter, we define the coefficient function

$$g(\varphi) = g_\kappa(\varphi) := (1 - \kappa)\varphi^2 + \kappa.$$

Hence, we replace the energy functional (2.1) by the regularized total energy [13, 14]

$$E_\varepsilon(q; u, \varphi) = \frac{1}{2} \left(g(\varphi) \mathbb{C}e(u), e(u) \right) - (q, u)_{\Gamma_N} + G_c \Gamma_\varepsilon(\varphi). \quad (2.3)$$

Within this setting, it is then required to find $(u(t), \varphi(t))$ minimizing the energy (2.3) subject to the irreversibility constraint

$$\varphi(t_2) \leq \varphi(t_1) \quad \forall t_1 \leq t_2.$$

Finally, it is common to discretize the evolution in time. To this end, we introduce an equidistant partition

$$0 = t_0 < t_1 < \dots < t_M = T,$$

with corresponding approximations $(u^i, \varphi^i)_{i=1}^M$ each minimizing the energy (2.3) subject to the constraint

$$\varphi^i \leq \varphi^{i-1}, \quad (2.4)$$

where φ^0 is some given initial phase field.

We remark that only the constraint $\varphi^i \leq \varphi^{i-1}$ is relevant for the minimization of (2.3) since the other bound $0 \leq \varphi^i$ is automatically satisfied, as we will see in Section 4.1.

2.4. Fracture Irreversibility and its Regularization. Due to the irreversibility constraint

$$\varphi^i \leq \varphi^{i-1}$$

on the fracture growth, optimization problems subject to such an evolution become mathematical programs with complementarity constraints (MPCC), see, e.g., [8, 43, 44].

Due to the complementarity condition, standard constraint qualifications for non-linear programs, like [47] or [53] can not be satisfied. Hence a zoo of different stationarity concepts has been introduced. For strong-stationarity, see, e.g., [44]. Unfortunately, in general, such a system is only necessary if a sufficiently large set of controls is admissible in the optimization problem, see, e.g., [49]. In all other cases, weaker concepts need to be considered to obtain stationarity systems that can sometimes be obtained as limits of relaxed formulations, see for instance [28, 29, 30]. For the error due to a finite element discretization of the obstacle problem, we refer to [40]. In contrast to the control of an obstacle problem additional difficulties arise due to the coupling of the phase-field variable with the elasticity problem.

Following a classical approach, see, e.g., [8], we regularize (2.4) to remove the inequality constraints involved in the fracture-propagation problem. Instead of the usual L^2 penalization approach, to ensure differentiability up to second order, we follow [39] and define the penalty for the irreversibility as

$$R(\varphi^{i-1}; \varphi^i) = \frac{1}{4} \|(\varphi^i - \varphi^{i-1})^+\|_{L^4}^4.$$

2.5. The Final Regularized Problem. In order to formulate the final forward problem, we introduce the spaces

$$V := H_D^1(\Omega; \mathbb{R}^2) \times H^1(\Omega), \quad Q := L^2(\Gamma_N).$$

Our final regularized time-discrete fracture problem for given initial data $(u^0, \varphi^0) \in V$ and given controls $q = (q^i)_{i=1}^M \in Q^M$ consists of finding $\mathbf{u} = (\mathbf{u}^i)_{i=1}^M = (u, \varphi) = ((u^i, \varphi^i)_{i=1}^M \in V^M$ solving the minimization problem

$$\min_{\mathbf{u}} E_\varepsilon^\gamma(q^i, \varphi^{i-1}; u^i, \varphi^i) := E_\varepsilon(q^i; u^i, \varphi^i) + \gamma R(\varphi^{i-1}; \varphi^i) \quad (\text{C}^\gamma)$$

for $i = 1, \dots, M$ and some given $\gamma > 0$.

3. The Optimization Problem. In this section, we formulate an optimal control problem in which the constraint is given by the regularized phase-field fracture problem from Section 2.5. We consider the following model problem in fracture propagation: for given $(u^0, \varphi^0) \in V$ with $0 \leq \varphi^0 \leq 1$, we wish to find $(q, \mathbf{u}) = (q, (u, \varphi)) \in (Q \times V)^M$ solving

$$\begin{aligned} \min_{q, \mathbf{u}} J(q, \mathbf{u}) &:= \frac{1}{2} \sum_{i=1}^M \|u^i - u_d^i\|^2 + \frac{\alpha}{2} \sum_{i=1}^M \|q^i\|_{\Gamma_N}^2 \\ \text{s.t. } \mathbf{u} &\text{ solves } (\text{C}^\gamma) \text{ given the data } q, \text{ for each } i = 1, \dots, M, \end{aligned} \quad (\text{P}^\gamma)$$

where $u_d \in (L^2(\Omega))^M$ is a given desired displacement. To obtain an infinite dimensional nonlinear program, we will furthermore replace (C^γ) by its first-order necessary optimality conditions. Formally, any minimizer $\mathbf{u} = (u, \varphi) \in V^M$ of (C^γ) satisfies the Euler-Lagrange equations

$$\begin{aligned} &(g(\varphi^i) \mathbb{C}e(u^i), e(v)) - (q^i, v)_{\Gamma_N} = 0, \\ &G_c \varepsilon (\nabla \varphi^i, \nabla \psi) - \frac{G_c}{\varepsilon} (1 - \varphi^i, \psi) \\ &+ (1 - \kappa) (\varphi^i \mathbb{C}e(u^i) : e(u^i), \psi) \\ &+ \gamma [(\varphi^i - \varphi^{i-1})^+]^3, \psi = 0 \end{aligned} \quad (\text{EL}^\gamma)$$

for any $(v, \psi) \in V$ and $i = 1, \dots, M$. However, since we relaxed the upper bound $\varphi^i \leq \varphi^{i-1}$ it is no longer clear, if all terms above are well-defined since it is not clear a priori whether $\varphi^i \in L^\infty(\Omega)$. We will, positively, answer this question in the following Section 4, utilizing results from [33] for damage models together with a Stampacchia-type cutoff argument.

With this we can further relax our problem, and obtain the regularized nonlinear problem, given $(u^0, \varphi^0) \in V$, $0 \leq \varphi^0 \leq 1$, to find $(q, \mathbf{u}) \in (Q \times V)^M$ solving

$$\begin{aligned} \min_{q, \mathbf{u}} J(q, \mathbf{u}) \\ \text{s.t. } (q^i, \mathbf{u}^i) &\text{ satisfy } (\text{EL}^\gamma) \text{ for each } i = 1, \dots, M. \end{aligned} \quad (\text{NLP}^\gamma)$$

While it is not clear that (P^γ) and (NLP^γ) have the same solutions, the regularity results obtained for (NLP^γ) will still be applicable to (P^γ) .

4. Existence of Solutions to (NLP^γ) . We proceed in two steps, starting by analyzing the lower level problem, before discussing the existence of solutions to (NLP^γ) .

4.1. The Phase-Field Model (EL $^\gamma$). Due to the fact, that we relaxed the constraint $\varphi^i \leq \varphi^{i-1}$ by a penalty approach, we can no longer assume $\varphi^i \in L^\infty$ as it is usually done in proving existence of minimizers to (2.3). The reason is that naively assuming minimal regularity asserted by the functional in (C $^\gamma$) the products of the variables, i.e., $\varphi^2 \mathbb{C}e(u) : e(u)$, are not in L^1 . Hence following ideas of Stampacchia [32], we will, temporarily, relax (C $^\gamma$) even further. Let $b > 0$ be an arbitrary given number and define

$$m = m_b : \mathbb{R} \rightarrow \mathbb{R}; \quad m(x) := \begin{cases} x & -b \leq x \leq b \\ P_{[-b-2, b+2]}(x) & \text{otherwise,} \end{cases}$$

where $P_{[-b-2, b+2]}$ is some smoothed projection onto $[-b-2, b+2]$ of which the precise definition is irrelevant as long as $m \in C^2$ with $0 \leq m' \leq 1$ and $m(\mathbb{R}) \subset [-b-2, b+2]$. With this, we define the regularized coefficient function

$$g_b(\varphi) = (1 - \kappa)m((\varphi^i)^2) + \kappa \in [\kappa, b+2].$$

We modify the cost functional in (C $^\gamma$) to include the cutoff function. Consequently, we consider the following family of problems

$$\begin{aligned} \min_{\mathbf{u}^i} E_\varepsilon^{\gamma, b}(q^i, \varphi^{i-1}; u^i, \varphi^i) &:= \frac{1}{2} \left(g_b(\varphi) \mathbb{C}e(u^i), e(u^i) \right) & (C^{\gamma, b}) \\ &- (q^i, u^i)_{\Gamma_N} + G_c \Gamma_\varepsilon(\varphi^i) + \gamma R(\varphi^{i-1}; \varphi^i), \end{aligned}$$

at each time-point $i = 1, \dots, M$. The idea of Stampacchia's method, in essence, is to prove that (C $^{\gamma, b}$) has all desired properties and, moreover, that for suitable $b \in \mathbb{R}$ the solutions of (C $^{\gamma, b}$) and (C $^\gamma$) coincide and thus our original problem inherits, among other properties, the boundedness of φ^i in L^∞ . Let us therefore start by discussing (C $^{\gamma, b}$), first.

LEMMA 4.1. *For any $i = 1, \dots, M$ it holds.*

1. *Given $q^i \in L^2(\Gamma_N)$ and $\varphi^{i-1} \in L^2(\Omega)$, (C $^{\gamma, b}$) has at least one solution $\bar{\mathbf{u}}^i$.*
2. *Further, any (local) minimizer $\bar{\mathbf{u}}^i$ of (C $^{\gamma, b}$) solves for all $(v, \psi) \in V$*

$$\begin{aligned} &\left(g_b(\varphi^i) \mathbb{C}e(u^i), e(v) \right) - (q^i, v)_{\Gamma_N} = 0 \\ &G_c \varepsilon (\nabla \varphi^i, \nabla \psi) + (1 - \kappa) (m'((\varphi^i)^2) \varphi^i \mathbb{C}e(u^i) : e(u^i), \psi) & (EL^{\gamma, b}) \\ &-\frac{G_c}{\varepsilon} (1 - \varphi^i, \psi) + \gamma ((\varphi^i - \varphi^{i-1})^+)^3, \psi = 0. \end{aligned}$$

3. *Finally, any solution $\mathbf{u}^i = (u^i, \varphi^i) \in V$ to (EL $^{\gamma, b}$) satisfies*
 - (a) *Assuming $\varphi^{i-1} \geq 0$ a.e. it follows $\varphi^i \geq 0$ a.e..*
 - (b) *There exists $p > 2$ and a constant $c_{b, \kappa}$ depending on b and κ (but not on \mathbf{u}^i), such that*

$$\|u^i\|_{1, p} \leq c_{b, \kappa} \|q^i\|.$$

- (c) *Assuming $\varphi^{i-1} \geq 0$ a.e., then*

$$\|\nabla \varphi^i\|^2 + \frac{\|\varphi^i\|^2}{2\varepsilon^2} \leq \frac{|\Omega|^2}{2\varepsilon^2}.$$

- (d) *Under the conditions above, we have*

$$0 \leq \varphi^i \leq 1.$$

Proof.

1. For any given $\varphi \in H^1(\Omega)$ it is $\kappa \leq g_b(\varphi) \leq b + 2$ and hence, by uniform convexity, there exists a unique minimizer $u = u(\varphi)$ of the elastic energy

$$u \mapsto \frac{1}{2}(g_b(\varphi)\mathbb{C}e(u), e(u)) - (q^i, u)_{\Gamma_N}.$$

It is thus sufficient to consider the reduced energy, compare, e.g., [33]

$$\min_{\varphi} \mathcal{E}_{\varepsilon}^{\gamma}(\varphi) := E_{\varepsilon}^{\gamma}(q^i, \varphi^{i-1}; u(\varphi), \varphi).$$

Utilizing the results of [26, Theorem 1.1], we obtain, for any $\varphi \in H^1(\Omega)$, the existence of $p > 2$ such that $u(\varphi) \in W^{1,p}(\Omega; \mathbb{R}^2) \cap H_D^1(\Omega; \mathbb{R}^2)$ and it holds

$$\|u(\varphi)\|_{1,p} \leq c_{b,\kappa} \|q^i\|.$$

Noticing that g_b satisfies the assumption [33, (2.10)] and the nonnegative penalty term $R(\varphi^{i-1}, \varphi)$ does not influence the statement, we can apply [33, Lemma 2.1] to see that the reduced energy satisfies

$$-\infty < c \leq \mathcal{E}_{\varepsilon}^{\gamma}(\varphi) \rightarrow \infty \quad (\|\varphi\|_{1,2} \rightarrow \infty).$$

Hence there exists $\varphi^i \in H^1(\Omega)$ and an H^1 -weakly convergent sequence $\varphi_k \rightharpoonup \varphi^i$ with

$$\mathcal{E}_{\varepsilon}^{\gamma}(\varphi_k) \rightarrow \inf_{\varphi} \mathcal{E}_{\varepsilon}^{\gamma}(\varphi) > -\infty.$$

By the compact embedding $H^1(\Omega) \subset L^4(\Omega)$, we can w.l.o.g. assume that $\varphi_k \rightarrow \varphi^i$ strongly in $L^4(\Omega)$, and hence convergence of $\gamma R(\varphi^{i-1}; \varphi_k) \rightarrow \gamma R(\varphi^{i-1}; \varphi^i)$ follows. By [33, Corollary 2.1] it follows that

$$\varphi \mapsto \mathcal{E}_{\varepsilon}^{\gamma}(\varphi) - \gamma R(\varphi^{i-1}; \varphi_k)$$

is weakly lower semi-continuous and hence

$$\inf_{\varphi} \mathcal{E}_{\varepsilon}^{\gamma}(\varphi) \leq \mathcal{E}_{\varepsilon}^{\gamma}(\varphi^i) \leq \lim_{k \rightarrow \infty} \mathcal{E}_{\varepsilon}^{\gamma}(\varphi_k) = \inf_{\varphi} \mathcal{E}_{\varepsilon}^{\gamma}(\varphi).$$

This completes the proof by setting $\mathbf{u}^i = (u(\varphi^i), \varphi^i)$.

2. We notice that for any $(v, \psi) \in V$ the mapping

$$S: \mathbb{R} \rightarrow \mathbb{R}; \quad s \mapsto E_{\varepsilon}^{\gamma}(q^i, \varphi^{i-1}; \mathbf{u}^i + s(v, \psi))$$

is well defined, differentiable and has a local minimizer at $s = 0$. This shows the assertion by consideration of the necessary optimality condition for a minimizer of S , i.e., $S'(0) = 0$.

3. (a) To show non-negativity of φ^i for the solutions of $(\text{EL}^{\gamma, b})$, we need to test the second equation in $(\text{EL}^{\gamma, b})$ with $\psi = \min(0, \varphi^i)$. We define the set

$$\Omega^- := \{x \in \Omega \mid \varphi^i(x) < 0\}$$

and obtain from (EL $^{\gamma,b}$)

$$\begin{aligned} 0 &= G_c \varepsilon \|\nabla \varphi^i\|_{\Omega^-}^2 + \frac{G_c}{\varepsilon} \|\varphi^i\|_{\Omega^-}^2 - \frac{G_c}{\varepsilon} (1, \varphi^i)_{\Omega^-} \\ &\quad + (1 - \kappa) (m'((\varphi^i)^2) (\varphi^i)^2 \mathbb{C}e(u^i), e(u^i))_{\Omega^-} \\ &\quad + \gamma ([(\varphi^i - \varphi^{i-1})^+]^3, \varphi^i)_{\Omega^-}. \end{aligned}$$

The first two terms are obviously non negative, and positive, if $|\Omega^-| > 0$. The third term satisfies $-(1, \varphi^i)_{\Omega^-} \geq 0$ by definition of Ω^- . The fourth term is nonnegative by our assumption on m' and \mathbb{C} . For the fifth (i.e., the final term), we notice, that by assumption on φ^{i-1}

$$\varphi^i \leq 0 \leq \varphi^{i-1} \quad \text{on } \Omega^-$$

and hence

$$([\varphi^i - \varphi^{i-1}]^+)^3, \varphi^i)_{\Omega^-} = 0.$$

This shows $|\Omega^-| = 0$ and hence the assertion $\varphi^i \geq 0$ a.e.

(b) As in the proof of 1. of this Lemma, the equation

$$(g_b(\varphi) \mathbb{C}e(u^i), e(v)) = (q^i, v)_{\Gamma_N}$$

implies the assertion utilizing [26, Proposition 1.2] noting that the estimates only depend on the lower and upper bounds on g_b and not the distribution of the intermediate values.

(c) We start by bounding the H^1 norm of φ^i . To this end, we test (EL $^{\gamma,b}$) with $\psi = \varphi^i$ and obtain

$$\begin{aligned} G_c \varepsilon \|\nabla \varphi^i\|^2 + \frac{G_c}{\varepsilon} \|\varphi^i\|^2 + \gamma ([(\varphi^i - \varphi^{i-1})^+]^3, \varphi^i) \\ + (1 - \kappa) (m'((\varphi^i)^2) (\varphi^i)^2 \mathbb{C}e(u^i), e(u^i)) \\ = \frac{G_c}{\varepsilon} (1, \varphi^i) \\ \leq \frac{G_c}{2\varepsilon} |\Omega|^2 + \frac{G_c}{2\varepsilon} \|\varphi^i\|^2. \end{aligned}$$

Since all terms on the left are non negative, noting that $\varphi^i \geq 0$, we deduce

$$\|\nabla \varphi^i\|^2 + \frac{\|\varphi^i\|^2}{2\varepsilon^2} \leq \frac{|\Omega|^2}{2\varepsilon^2}.$$

(d) To see the statement, we test (EL $^{\gamma,b}$) with $\psi = (\varphi^i - 1)^+ = \max(0, \varphi^i - 1)$ and obtain

$$\begin{aligned} 0 &= G_v \varepsilon \|\nabla (\varphi^i - 1)^+\|^2 + \frac{G_c}{\varepsilon} \|(\varphi^i - 1)^+\|^2 + \gamma ([(\varphi^i - \varphi^{i-1})^+]^3, (\varphi^i - 1)^+) \\ &\quad + (1 - \kappa) (m'((\varphi^i)^2) \varphi^i \mathbb{C}e(u^i) : e(u^i), (\varphi^i - 1)^+). \end{aligned}$$

Noticing that all summands are non negative, the assertion follows analogously to part (a).

□

Now, choosing $b \geq 1$ in the last Lemma allows to transfer these results to (C^γ) .

COROLLARY 4.2.

1. Given $q^i \in L^2(\Gamma_N)$ and $\varphi^{n-1} \in L^2(\Omega)$ with $\varphi^{i-1} \geq 0$ there exists at least one solution \mathbf{u}^i of (EL^γ) .
2. Further, any solution $\mathbf{u}^i = (u^i, \varphi^i)$ to (EL^γ) satisfies
 - (a) $0 \leq \varphi^i \leq 1$ a.e.
 - (b) There exists $p > 2$ and a constant c_κ depending on κ , such that

$$\|u^i\|_{1,p} \leq c_\kappa \|q^i\|.$$

(c) It holds

$$\|\nabla \varphi^i\|^2 + \frac{\|\varphi^i\|^2}{2\varepsilon^2} \leq \frac{|\Omega|^2}{2\varepsilon^2}.$$

Proof.

1. The existence of at least one solution follows by Lemma 4.1 taking $b \geq 1$ since then $g_b(\varphi^i) = g(\varphi^i)$ and $m'((\varphi^i)^2) = 1$ for any solution to $(EL^{\gamma,b})$ and hence any such solution solves (EL^γ) as well.
2. (a) The proof of Lemma 4.1 3.(a) and 3.(d) can be repeated to yield the desired bounds $0 \leq \varphi^i \leq 1$.
- (b) The proof of Lemma 4.1 3.(b) can be applied, noticing that the constant depends on the upper and lower bound of the coefficient, i.e., $\kappa \leq g(\varphi) \leq 1$, only.
- (c) The proof of Lemma 4.1 3.(c) carries over to the present setting as well.

□

4.2. The Problem (NLP^γ) . We can now finalize the existence of solutions to (NLP^γ) .

THEOREM 4.3. *There exists at least one global minimizer $(q, \mathbf{u}) \in (Q \times V)^M$ to (NLP^γ) .*

Proof. The proof is almost straight forward. Since $J(q, \mathbf{u}) \geq 0$ there exists a minimizing sequence (q_k, \mathbf{u}_k) satisfying (EL^γ) , i.e., $J(q_k, \mathbf{u}_k) \rightarrow \inf_{q, \mathbf{u}} J(q, \mathbf{u})$. The corresponding control q_k is bounded in Q^M and hence there exists a weakly convergent subsequence, w.l.o.g denoted by q_k , with limit q_∞ . By Corollary 4.2, namely 2.(b) and 2.(c) therein, the sequence (u_k, φ_k) is bounded in $(W^{1,p}(\Omega; \mathbb{R}^2) \times H^1(\Omega))^M$ and consequently w.l.o.g. $u_k \rightharpoonup u_\infty$ in $W^{1,p}(\Omega; \mathbb{R}^2)^M$ and $\varphi_k \rightharpoonup \varphi_\infty$ in $H^1(\Omega)^M$. To see that the limit satisfies the elasticity equation in (EL^γ) , note that due to the compact embedding $H^1(\Omega) \subset L^p(\Omega)$ for any $p < \infty$

$$g(\varphi_k^i) \mathbb{C}e(u_k^i) \rightharpoonup g(\varphi_\infty^i) \mathbb{C}e(u_\infty^i)$$

in $L^2(\Omega; \mathbb{R}^{2 \times 2})$ holds, since $g(\varphi_k^i)$ converges strongly. To see that the limiting phase-field φ_∞ satisfies the equation, we notice, that by Corollary 4.2 the phase-field satisfies $\varphi_k^i \leq 1$ and hence we can also consider $(EL^{\gamma,b})$, because it coincides with (EL^γ) in the relevant points but satisfies the conditions in [33]. Thus by [33, Corollary 2.1], we obtain

$$\begin{aligned} G_c \varepsilon (\nabla \varphi_k^i, \nabla \cdot) + (1 - \kappa) (\varphi_k^i \mathbb{C}e(u_k^i) : e(u_k^i), \cdot) - \frac{G_c}{\varepsilon} (1 - \varphi_k^i, \cdot) \\ \rightharpoonup G_c \varepsilon (\nabla \varphi_\infty^i, \nabla \cdot) + (1 - \kappa) (\varphi_\infty^i \mathbb{C}e(u_\infty^i) : e(u_\infty^i), \cdot) - \frac{G_c}{\varepsilon} (1 - \varphi_\infty^i, \cdot) \end{aligned}$$

weakly in $H^1(\Omega)^*$. For the remaining term $\gamma[(\varphi_k^i - \varphi_k^{i-1})^+]^3$ the compact embedding $H^1(\Omega) \subset L^6(\Omega)$ gives convergence in L^2 and consequently the pair $\mathbf{u}_\infty = (u_\infty, \varphi_\infty)$ solves (EL^γ) .

Hence $(q_\infty, \mathbf{u}_\infty)$ is feasible for (NLP^γ) . Weak lower semicontinuity of J shows that

$$J(q_\infty, \mathbf{u}_\infty) \leq \inf_{q, \mathbf{u}} J(q, \mathbf{u})$$

and thus the $(q_\infty, \mathbf{u}_\infty)$ is a global minimizer, setting $(q, \mathbf{u}) = (q_\infty, \mathbf{u}_\infty)$. \square

COROLLARY 4.4. *Any minimizer (q, \mathbf{u}) of (NLP^γ) satisfies the additional regularity $\mathbf{u} \in (W^{1,p}(\Omega; \mathbb{R}^2) \times L^\infty(\Omega))^M$ for some $p > 2$. More precisely for any $i = 1, \dots, M$ it holds $0 \leq \varphi^i \leq 1$ and $\|u^i\|_{1,p} \leq c_\kappa \|q^i\|$.*

Proof. This is an immediate consequence of Corollary 4.2. \square

5. The Linearized Problem. In order to discuss first order necessary optimality conditions, as well as the potential approximation of (NLP^γ) by a sequence of linear-quadratic problems, let $(q_k, \mathbf{u}_k) = (q_k, u_k, \varphi_k) \in (Q \times V)^M$ be a given point. Considering the regularity of solutions to (EL^γ) , we assume $q_k \in Q^M$, and $(u_k, \varphi_k) \in (V \cap (W^{1,p}(\Omega; \mathbb{R}^2) \times L^\infty(\Omega)))^M$.

The linearized problem to (EL^γ) consists, for given $q \in Q^M$ and $\varphi^0 := 0$, of finding $\mathbf{u} = (u, \varphi) \in V^M$ such that for any $i = 1, \dots, M$ and $(v, \psi) \in V$

$$\begin{aligned} & \left(g(\varphi_k^i) \mathbb{C}e(u^i), e(v) \right) \\ & + 2(1 - \kappa)(\varphi_k^i \mathbb{C}e(u_k^i) \varphi^i, e(v)) = (q^i, v)_{\Gamma_N} \\ & G_c \varepsilon (\nabla \varphi^i, \nabla \psi) + \frac{G_c}{\varepsilon} (\varphi^i, \psi) \\ & + (1 - \kappa)(\varphi^i \mathbb{C}e(u_k^i) : e(u_k^i), \psi) \\ & + 3\gamma([\varphi_k^i - \varphi_k^{i-1}]^+)^2 \varphi^i, \psi \\ & + 2(1 - \kappa)(\varphi_k^i \mathbb{C}e(u_k^i) : e(u^i), \psi) = 3\gamma([\varphi_k^i - \varphi_k^{i-1}]^+)^2 \varphi^{i-1}, \psi. \end{aligned} \tag{EL}_{\text{lin}}^\gamma$$

Existence of Solutions to $(\text{EL}_{\text{lin}}^\gamma)$. We now discuss the properties of the linearized equation $(\text{EL}_{\text{lin}}^\gamma)$.

LEMMA 5.1. *For any given $(u_k, \varphi_k) \in (V \cap (W^{1,p}(\Omega; \mathbb{R}^2) \times L^\infty(\Omega)))^M$ with $p > 2$ and $q_k \in Q^M$ the linear operators $A_i : V \rightarrow V^*$ corresponding to $(\text{EL}_{\text{lin}}^\gamma)$ defined by*

$$\begin{aligned} \langle A_i(u^i, \varphi^i), (v, \psi) \rangle_{V, V^*} & := a_i(u^i, \varphi^i; v, \psi) \\ & := \left(g(\varphi_k^i) \mathbb{C}e(u^i), e(v) \right) + 2(1 - \kappa)(\varphi_k^i \mathbb{C}e(u_k^i) \varphi^i, e(v)) \\ & + G_c \varepsilon (\nabla \varphi^i, \nabla \psi) + \frac{G_c}{\varepsilon} (\varphi^i, \psi) + (1 - \kappa)(\varphi^i \mathbb{C}e(u_k^i) : e(u_k^i), \psi) \\ & + 3\gamma([\varphi_k^i - \varphi_k^{i-1}]^+)^2 \varphi^i, \psi + 2(1 - \kappa)(\varphi_k^i \mathbb{C}e(u_k^i) : e(u^i), \psi) \end{aligned}$$

are Fredholm of index zero.

Proof. Since $p > 2$, we can find $r \in (2, \infty)$ such that $\frac{1}{p} + \frac{1}{2} + \frac{1}{r} = 1$. By embedding theorems, there exists $0 < s < 1$, such that $H^s(\Omega) \subset L^r(\Omega)$ compactly.

Then continuity of a_i on $V \times V$ follows

$$\begin{aligned}
a_i(u^i, \varphi^i; v, \psi) &\leq c\|u^i\|_{1,2}\|v\|_{1,2} + c\|\varphi^i\|_{0,r}\|v\|_{1,2} \\
&\quad + c\|\varphi^i\|_{1,2}\|\psi\|_{1,2} + c\|\varphi^i\|_{0,r}\|\psi\|_{0,r} + c\|\varphi^i\|_{0,r}\|\psi\|_{0,r} \\
&\quad + c_\gamma\|\varphi^i\|_{1,2}\|\psi\|_{1,2} + c\|u^i\|_{1,2}\|\psi\|_{0,r} \\
&\leq c(\|u^i\|_{1,2}\|v\|_{1,2} + \|\varphi^i\|_{1,2}\|v\|_{1,2} + \|\varphi^i\|_{1,2}\|\psi\|_{1,2} + \|u^i\|_{1,2}\|\psi\|_{1,2}) \\
&\leq c\|(u^i, \varphi^i)\|_V\|(v, \psi)\|_V,
\end{aligned}$$

with generic constants c depending on $(u_k^i, \varphi_k^i) \in W^{1,p}(\Omega; \mathbb{R}^2) \times L^\infty(\Omega)$. To derive a lower bound, we notice that the only possibly non-positive terms are the two starting with $2(1 - \kappa)$ and we deduce, using Korn's inequality

$$\begin{aligned}
a_i(u^i, \varphi^i; u^i, \varphi^i) &\geq c\|u^i\|_{1,2}^2 + \|\varphi^i\|_{1,2}^2 - c\|\varphi^i\|_{0,r}\|u^i\|_{1,2} \\
&\geq c\|u^i\|_{1,2}^2 + \|\varphi^i\|_{1,2}^2 - c\|\varphi^i\|_{s,2}^2.
\end{aligned}$$

Consequently $a_i(\cdot, \cdot) + c(\cdot, \cdot)_{s,2}$ is coercive on $V \times V$ and thus invertible, and in particular Fredholm of index zero, by the Lax-Milgram lemma. Since $H^1(\Omega) \subset H^s(\Omega)$ is compact, we deduce that the mapping $A_i: V \rightarrow V^*$ given by $(u^i, \varphi^i) \mapsto A_i(u^i, \varphi^i) = a_i(u^i, \varphi^i; \cdot)$ is Fredholm of index zero as well, see, e.g., [52, Theorem 12.8]. \square

LEMMA 5.2. *Under the assumptions of Lemma 5.1, any element $(u^i, \varphi^i) \in \ker(A_i) \subset V$ satisfies the additional regularity $(u^i, \varphi^i) \in V \cap (W^{1,p}(\Omega; \mathbb{R}^2) \times L^\infty(\Omega))$, with $p > 2$ as in Corollary 4.4*

Proof. Consider $(u^i, \varphi^i) \in \ker(A_i)$, i.e.,

$$a_i(u^i, \varphi^i; v, \psi) = 0 \quad \forall (v, \psi) \in V.$$

First of all, we notice, that the linearized phase-field φ^i satisfies

$$\begin{aligned}
G_c \varepsilon (\nabla \varphi^i, \nabla \psi) + \frac{G_c}{\varepsilon} (\varphi^i, \psi) \\
= -(1 - \kappa) (\varphi^i \mathbb{C} e(u_k^i) : e(u_k^i), \psi) - 3\gamma [(\varphi_k^i - \varphi_k^{i-1})^+]^2 \varphi^i, \psi \\
- 2(1 - \kappa) (\varphi_k^i \mathbb{C} e(u_k^i) : e(u^i), \psi).
\end{aligned}$$

With the definition of r as in the proof of Lemma 5.1, it is $\varphi^i \in L^r(\Omega)$ and $\frac{1}{p} + \frac{1}{r} = \frac{1}{2}$. Let r' be given such that $1 = \frac{1}{r} + \frac{1}{r'} = \frac{1}{r} + (\frac{1}{r} + \frac{2}{p})$, then $1 < r' < 2$. As a consequence, the right hand side of the equation above is an element in $L^{r'}(\Omega)$. To see this, we calculate

$$\begin{aligned}
\|\varphi^i \mathbb{C} e(u_k^i) : e(u_k^i)\|_{0,r'} &\leq c\|\varphi^i\|_{0,r}\|e(u_k^i)\|_{0,p}^2, \\
\|[(\varphi_k^i - \varphi_k^{i-1})^+]^2 \varphi^i\|_{0,r'} &\leq c\|[(\varphi_k^i - \varphi_k^{i-1})^+]^2 \varphi^i\|_{0,r} \leq c\|(\varphi_k^i - \varphi_k^{i-1})^+\|_{0,\infty}^2 \|\varphi^i\|_{0,r}, \\
\|\varphi_k^i \mathbb{C} e(u_k^i) : e(u^i)\|_{0,r'} &\leq c\|\varphi_k^i\|_{0,\infty}\|e(u_k^i)\|_{0,p}\|e(u^i)\|.
\end{aligned}$$

Utilizing elliptic regularity it follows that $\varphi^i \in H^1(\Omega) \cap L^\infty(\Omega)$.

Now, we can continue to derive the improved regularity of u^i . To this end, we notice, that u^i solves

$$\left(g(\varphi_k^i) \mathbb{C} e(u^i), e(v) \right) = -2(1 - \kappa) (\varphi_k^i \mathbb{C} e(u_k^i) \varphi^i, e(v)).$$

The right hand side satisfies

$$\|\varphi_k^i \mathbb{C}e(u_k^i) \varphi^i\|_{0,p} \leq c \|\varphi_k^i\|_{0,\infty} \|u_k^i\|_{1,p} \|\varphi^i\|_{0,\infty}$$

and thus $(\varphi_k^i \mathbb{C}e(u_k^i) \varphi, e(\cdot))$ defines an element in $W^{-1,p}(\Omega; \mathbb{R}^2) = \left(W^{1,p'}(\Omega; \mathbb{R}^2)\right)^*$, utilizing again [26, Theorem 1.1], we conclude that $u \in W^{1,p}(\Omega; \mathbb{R}^2)$. \square

REMARK 5.3. *Utilizing the above regularity provided by Lemma 5.2, we can now define the scalar product $(\cdot, \cdot)_{\mathbb{C}} = (\mathbb{C}\cdot, \cdot)$ and corresponding norm $\|\cdot\|_{\mathbb{C}}$. The above regularity shows, that the norms $\|\varphi^i e(u_k^i)\|_{\mathbb{C}}$ and $\|\varphi_k^i e(u^i)\|_{\mathbb{C}}$ are finite for all $(u^i, \varphi^i) \in \ker A_i$. Consequently, we can now provide an improved lower bound utilizing the parallelogram identity for the above scalar product*

$$\begin{aligned} 0 &= a_i(u^i, \varphi^i; u^i, \varphi^i) \\ &= (1 - \kappa)((\varphi_k^i)^2 \mathbb{C}e(u^i), e(u^i)) + \kappa(\mathbb{C}e(u^i), e(u^i)) + 2(1 - \kappa)(\varphi_k^i \mathbb{C}e(u_k^i) \varphi^i, e(u^i)) \\ &\quad + G_c \varepsilon (\nabla \varphi^i, \nabla \varphi^i) + \frac{G_c}{\varepsilon} (\varphi^i, \varphi^i) + (1 - \kappa)(\varphi^i \mathbb{C}e(u_k^i) : e(u_k^i), \varphi^i) \\ &\quad + 3\gamma[(\varphi_k^i - \varphi_k^{i-1})^+]^2 \varphi^i, \varphi^i) + 2(1 - \kappa)(\varphi_k^i \mathbb{C}e(u_k^i) : e(u^i), \varphi^i) \\ &\geq \kappa \|e(u^i)\|_{\mathbb{C}}^2 + (1 - \kappa) \|\varphi_k^i e(u^i)\|_{\mathbb{C}}^2 + 2(1 - \kappa)(\varphi_k^i e(u^i), \varphi^i e(u_k^i))_{\mathbb{C}} \\ &\quad + c \|\varphi^i\|_{1,2}^2 + (1 - \kappa) \|\varphi^i e(u_k^i)\|_{\mathbb{C}}^2 + 2(1 - \kappa)(\varphi_k^i e(u^i), \varphi^i e(u_k^i))_{\mathbb{C}} \\ &\geq \kappa \|e(u^i)\|_{\mathbb{C}}^2 + c \|\varphi^i\|_{1,2}^2 + (1 - \kappa) \|\varphi_k^i e(u^i) + \varphi^i e(u_k^i)\|_{\mathbb{C}}^2 + 2(1 - \kappa)(\varphi_k^i e(u^i), \varphi^i e(u_k^i))_{\mathbb{C}}. \end{aligned}$$

REMARK 5.4. *From the previous Remark 5.3, we immediately assert, that for sufficiently small $\|u_k^i\|_{1,p}$, $\|\varphi_k^i\|_{0,\infty}$, the mixed term can be absorbed into the squared norms and, consequently, for such (u_k, φ_k) , we have $\ker A_i = \{0\}$. Indeed, this would already be clear from the proof of Lemma 5.1, but the condition provided by Remark 5.3 is tighter.*

COROLLARY 5.5. *For any given $(u_k, \varphi_k) \in (V \cap (W^{1,p}(\Omega; \mathbb{R}^2) \times L^\infty(\Omega)))^M$ and $q_k \in Q^M$ the linear operators $\mathcal{A}: V^M \rightarrow (V^*)^M$ defined by*

$$\begin{pmatrix} A_1 & & & & \\ B_2 & A_2 & & & \\ & & \ddots & \ddots & \\ & & & B_M & A_M \end{pmatrix}$$

with $A_i: V \rightarrow V^*$ as in Lemma 5.1 and $B_i = 3\gamma[(\varphi_k^i - \varphi_k^{i-1})^+]^2$, are Fredholm of index zero.

Proof. By Lemma 5.1 the diagonal is Fredholm, and the off-diagonal B_i are compact as a mapping $V \rightarrow V^*$. Thus the assertion follows by [52, Theorem 12.8]. \square

6. First Order Necessary Conditions for (NLP^γ) . We can now state the necessary optimality conditions for (NLP^γ) .

THEOREM 6.1. *Let $(\bar{q}, \bar{\mathbf{u}}) \in (Q \times V)^M$ be a minimizer of (NLP^γ) , such that $\ker \mathcal{A} = \{0\}$, with \mathcal{A} as defined in Corollary 5.5 in the point $(q_k, \mathbf{u}_k) = (\bar{q}, \bar{\mathbf{u}})$. Then*

there exists $\bar{\mathbf{z}} = (\bar{z}, \bar{\zeta}) \in V^M$ such that

$$\begin{aligned} (\bar{q}, \bar{\mathbf{u}}) &\text{ satisfy } (\text{EL}^\gamma) \\ \langle \mathcal{A}^* \bar{\mathbf{z}}, \varphi \rangle &= \sum_{i=1}^M (\bar{u}^i - u_d^i, \varphi^i) \quad \forall \varphi \in V^M \\ \alpha \sum_{i=1}^M (\bar{q}^i, \delta q^i)_{\Gamma_N} &= - \sum_{i=1}^M (\bar{z}^i, \delta q^i)_{\Gamma_N} \quad \forall \delta q \in Q^M. \end{aligned}$$

Proof. By Corollary 5.5 \mathcal{A} is Fredholm, since $\ker \mathcal{A} = \{0\}$ \mathcal{A} is an isomorphism, and so is its dual \mathcal{A}^* . Consequently, the linearized constraint $(\text{EL}_{\text{lin}}^\gamma)$ is surjective as a mapping $(Q \times V)^M \rightarrow (V^*)^M$ and the existence of $\bar{\mathbf{z}}$ follows by standard results on the existence of Lagrange multipliers, see, e.g., [53, Theorem 4.1.(a)]. \square

7. Quadratic Approximations to (NLP^γ) . We aim to approximate (NLP^γ) by a linear-quadratic problem in a given point $(q_k, \mathbf{u}_k) = (q_k, u_k, \varphi_k)$. Considering the regularity of solutions to (EL^γ) , we assume $q_k \in Q^M$, and $(u_k, \varphi_k) \in (V \cap (W^{1,p}(\Omega; \mathbb{R}^2) \times L^\infty(\Omega)))^M$.

In order to keep the notation short, we introduce the (compact) operator $\mathcal{B}: Q^M \rightarrow (V^*)^M$ for the control action as follows

$$\langle \mathcal{B}q, (v, \psi) \rangle_{(V^*)^M, V^M} := \sum_{i=1}^M (q^i, v^i)_{\Gamma_N}. \quad (7.1)$$

By standard reformulations, we obtain the quadratic problem, up to a fixed additive constant in the cost functional,

$$\begin{aligned} \min_{(q, \mathbf{u})} J_{\text{lin}}(q, \mathbf{u}) &:= \frac{1}{2} \sum_{i=1}^M \|u^i - (u_d^i - u_k^i)\|^2 + \frac{\alpha}{2} \sum_{i=1}^M \|q^i + q_k^i\|_{\Gamma_N}^2 \\ \text{s.t. } (q, \mathbf{u}) &\text{ satisfy } (\text{EL}_{\text{lin}}^\gamma), \text{ i.e., } \mathcal{A}\mathbf{u} = \mathcal{B}q, \end{aligned} \quad (\text{QP}^\gamma)$$

where \mathcal{A} is given in Corollary 5.5 and \mathcal{B} in (7.1).

7.1. Existence of Solutions to (QP^γ) . **THEOREM 7.1.** *For any given $(u_k, \varphi_k) \in (V \cap (W^{1,p}(\Omega; \mathbb{R}^2) \times L^\infty(\Omega)))^M$ and $q_k \in Q^M$ the problem (QP^γ) has a unique solution $(\bar{q}, \bar{\mathbf{u}}) \in Q^M \times V^M$.*

Proof. It is immediate that, a pair (q, \mathbf{u}) satisfies $(\text{EL}_{\text{lin}}^\gamma)$ if and only if

$$\mathcal{A}\mathbf{u} = \mathcal{B}q \quad \text{in } (V^*)^M$$

with \mathcal{A} as defined in Corollary 5.5 and \mathcal{B} as in (7.1). Now, by Corollary 5.5, \mathcal{A} is Fredholm and consequently, see, e.g., [52, Theorem 12.2], has closed range. Moreover, since the codimension of the image of \mathcal{A} is finite the intersection $\mathcal{A}(V^M) \cap \mathcal{B}(Q^M)$ is non empty. Clearly J_{lin} is bounded below, and we can pick a minimizing sequence $(q_{(k)}, \mathbf{u}_{(k)})$ satisfying $\mathcal{A}\mathbf{u}_{(k)} = \mathcal{B}q_{(k)}$. Due to the coercivity of J_{lin} the sequence is bounded and, possibly selecting a subsequence, there is a weak limit $q_{(k)} \rightharpoonup q_{(\infty)}$ in Q^M . By compactness, $\mathcal{B}q_{(k)} \rightarrow \mathcal{B}q_{(\infty)}$ in $(V^*)^M$.

Since \mathcal{A} is Fredholm, $\dim \ker \mathcal{A} < \infty$ and consequently, we can decompose $V^M = \ker \mathcal{A} \oplus V^M / \ker \mathcal{A}$. Correspondingly, we split the sequence $\mathbf{u}_{(k)} = \mathbf{u}_{(k)}^{\ker} + \mathbf{u}_{(k)}^0$. Then \mathcal{A} induces an isomorphism as a mapping $\mathcal{A}: V^M / \ker \mathcal{A} \rightarrow \mathcal{A}(V^M)$ and consequently

$$\mathbf{u}_{(k)}^0 = \mathcal{A}^{-1} \mathcal{B}q_{(k)} \rightarrow \mathcal{A}^{-1} \mathcal{B}q_{(\infty)} = \mathbf{u}_{(\infty)}^0.$$

Moreover, since J_{lin} is bounded along its minimizing sequence, $\|\mathbf{u}_{(k)}^{\text{ker}}\|$ is bounded, and since $\ker \mathcal{A}$ is finite dimensional, possibly selecting a subsequence, there exists a limit $\mathbf{u}_{(k)}^{\text{ker}} \rightarrow \mathbf{u}_{(\infty)}^{\text{ker}} \in \ker \mathcal{A}$. By continuity of \mathcal{A} it is

$$\mathcal{A}\mathbf{u}_{(\infty)} = \mathcal{B}q_{(\infty)},$$

and by weak lower semicontinuity

$$J_{\text{lin}}(q_{(\infty)}, \mathbf{u}_{(\infty)}) \leq \inf_{(q, \mathbf{u})} J_{\text{lin}}(q, \mathbf{u}).$$

Uniqueness follows, since J_{lin} is strictly convex on $Q^M \times V^M$. \square

7.2. Necessary (& Sufficient) Optimality Conditions. To conclude the discussion of the quadratic approximations, we note that we can give necessary, and due to convexity also sufficient, first order optimality conditions.

THEOREM 7.2. *For any given $(u_k, \varphi_k) \in (V \cap (W^{1,p}(\Omega; \mathbb{R}^2) \times L^\infty(\Omega)))^M$ and $q_k \in Q^M$ let $(\bar{q}, \bar{\mathbf{u}}) \in Q^M \times V^M$ be a solution to (QP^γ) . Then there exists a Lagrange multiplier, $\bar{\mathbf{z}} = (\bar{z}, \bar{\zeta}) \in V^M$, such that the system*

$$\begin{aligned} \mathcal{A}\bar{\mathbf{u}} &= \mathcal{B}\bar{q} && \text{in } (V^*)^M, \\ \mathcal{A}^*\bar{\mathbf{z}} &= \bar{\mathbf{u}} - (u_d - u_k) && \text{in } (V^*)^M, \\ \alpha(\bar{q} - q_k) + \bar{z} &= 0 && \text{on } \Gamma_N \end{aligned} \quad (\text{KKT}^\gamma)$$

is satisfied where \mathcal{A} is given in Corollary 5.5, \mathcal{A}^* denotes its adjoint, \mathcal{B} is given by (7.1), and the, compact, embedding $(L^2)^M \subset (V^*)^M$ is used without special notation for the right hand side of the adjoint equation. Due to the convexity of (QP^γ) , any triplet $(\bar{q}, \bar{\mathbf{u}}, \bar{\mathbf{z}}) \in Q^M \times V^M \times V^M$ solving (KKT^γ) gives rise to a solution of (QP^γ) .

Proof. We notice that the equality constraint in (QP^γ) is linear, and consequently a constraint qualification is given and the result is a consequence of Farkas'-Lemma, see, e.g., [17, Theorem 10] for its generalization to infinite dimensions. \square

8. Numerical Illustration. In this final section, we discuss a prototype test in order to substantiate our theoretical advancements. The setup is to employ a control q on the top boundary of a two-dimensional square domain, acting in normal direction only, in order to steer the solution towards a manufactured solution u_D defined in the entire domain. The computations are performed with DOpElib [21, 22] utilizing the deal.II finite element library [6, 7]. The discretized optimization problem is solved by a globalized Newton's method for the reduced optimization problem, i.e., the optimization problem is transformed into an unconstrained problem via elimination of the equality constraint due to the choice of a specific solution of the discretized fracture problem. In our numerical example this solution was found utilizing a globalized Newton's method for the discretized Euler-Lagrange equation (EL^γ) . If instead of just finding any stationary point of this equation a, local, minimizer of the phase-field energy (C^γ) is desired, alternating direction methods [12, 13, 15, 38] are typically used for the solution of (EL^γ) . Recently, there have also been efforts to employ monolithic algorithms such as via a convexification trick [25] and a line-search assisted Newton method [20]. In fact the latter paper provides comparisons with alternating directions and shows that a monolithic approach is not only more robust but also more efficient.

Our findings indicate that the QP-approximations discussed above can be used to obtain a (locally) fast convergent Newton (SQP) Algorithm.

The domain is given by $\Omega := (-1, 1)^2$ in which a horizontal fracture is prescribed. The initial value for φ^0 is taken such that $\varphi^0 = 0$ on $(-0.1 - h, 0.1 + h) \times (-h, h) \subset \Omega$ (see Figure 8.1), where h denotes the diameter of the elements. The boundary is divided into three parts $\partial\Omega := \Gamma_N \cup \Gamma_D \cup \Gamma_{\text{free}}$ corresponding to the control boundary Γ_N , the Dirichlet boundary Γ_D , and the rest, where natural boundary conditions for the displacement are attained. These boundary parts are given by

$$\Gamma_N = \{(x, 1) \mid -1 \leq x \leq 1\} \quad \text{and} \quad \Gamma_{\text{free}} = \{(x, y) \mid x \in \{\pm 1\}; -1 \leq y \leq 1\}.$$

On Γ_D , we prescribe the Dirichlet values $u = 0$.

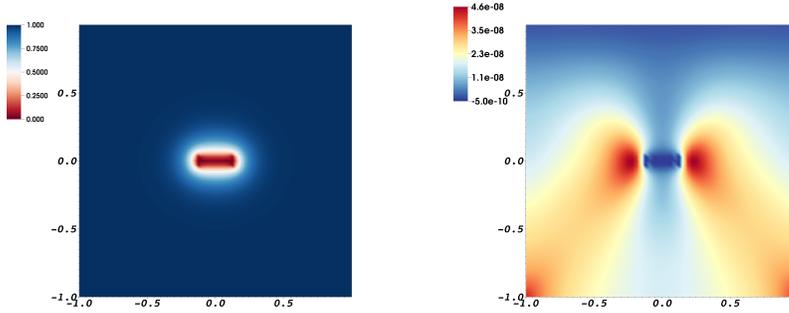


FIG. 8.1. Initial fracture (in red) and final adjoint phase-field after four Newton iterations at $M = 5$.

The cost functional is given by

$$J(q, \mathbf{u}) := \frac{1}{2} \sum_{i=1}^M \|u^i - u_d^i\|^2 + \frac{\alpha}{2} \|q + q_d\|_{\Gamma_N}^2$$

s.t. (q, \mathbf{u}) satisfying (EL^γ) ,

where $u_d^i = 0.001(y + 1)$ for all $i = 1, \dots, M$, $\alpha = 10^{-10}$ and a control acting on Γ_N but being the same in all time-steps, i.e. $q^i = q$ for all $i = 1, \dots, M$, and $q_d \equiv 50$. Moreover, $\mathbf{u}^0 = (0; \varphi^0)$ with φ^0 as depicted in Figure 8.1.

Furthermore, the phase-field regularization parameter is chosen as $\varepsilon = 2h = 0.088$ where $h = 0.0442$ is the element diameter of the mesh for the finite element discretization used for the computations. The bulk regularization parameter is $\kappa = 10^{-10}$.

The penalization parameter is $\gamma = 10^8$, the fracture energy release rate is $G_c = 1.0$, Young's modulus is $E = 10^6$ and Poisson's ratio is $\nu = 0.2$. The initial mesh is six times globally refined and 5 loading steps, i.e., $M = 5$, are performed. The spatial discretization is done using standard Q_1 finite elements for all unknowns.

Our findings are summarized in the following. The initial value of the cost functional is $J_{\text{initial}} = 1.247 \times 10^{-5}$ that is obtained by employing the initial control $q \equiv 10$ on Γ_N . In this particular setting, the initial residual of the Newton iteration is small; namely 7.46×10^{-9} . This starting residual is taken as 1 in the relative residual, which is plotted in Table 8.1.

TABLE 8.1
Results of the nonlinear optimization iterations.

Newton iter.	N-linear iter.	Newton residual (rel.)	$J[\times 10^{-5}]$	q_{max} on Γ_N
0	-	1.00×10^0	1.2470	10
1	14	3.57×10^{-2}	1.0487	87
2	11	1.23×10^{-3}	1.0469	84
3	2	3.94×10^{-4}	1.0469	84
4	2	1.27×10^{-4}	1.0469	84

Furthermore, Table 8.1 shows the iteration history of the Newton steps performed during the solution of the optimization problem. At each step, the Newton residual, the cost functional J and $q_{max} = \max_{\Gamma_N} |q|$ are provided. We observe that the algorithm is convergent, the convergence slows down to a linear rate in the later iterations as it has to be expected since the QP-subproblems are solved only up to an accuracy proportional to the norm of the optimization residual, and consequently only very few, i.e., two, iterations of the linear solver are performed in these Newton steps.

Illustrations of the solutions are provided in the Figures 8.1–8.3 displaying the primal and adjoint solutions.

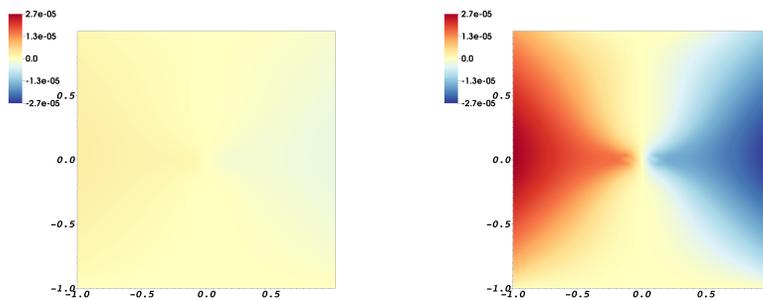


FIG. 8.2. *Initial x-displacement field and final x-displacement field after four Newton iterations at $M = 5$.*

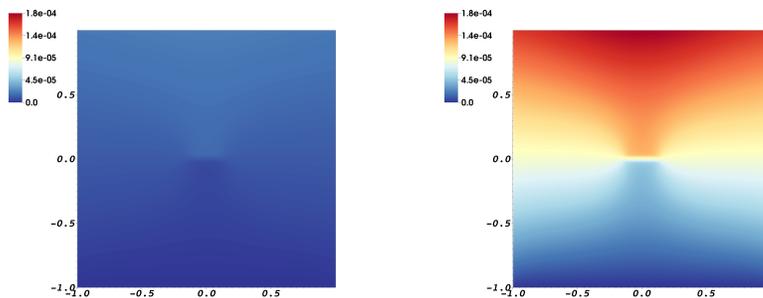


FIG. 8.3. *Initial y-displacement field and final y-displacement field after four Newton iterations at $M = 5$.*

Although the final fracture has not visibly changed compared to the initial value the adjoint phase field in Figure 8.1 shows the strong influence of the presence of the fracture onto the optimization. The displacements shown in Figure 8.2 and 8.3 show the desired behavior. It should be noted, that the color-scale in 8.2 and 8.3 is adjusted to the size of the displacement in the last Newton step, as it is visible from these pictures the initial displacement is severely smaller and almost invisible in this scale.

Indeed the y -displacement is almost almost uniform on Γ_N . In contrast to the behavior of a purely linear elastic body the fracture and its influence are clearly visible in the final x and y displacements.

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