Dual-weighted residual adaptivity for phase-field fracture propagation

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In this study, we consider phase-field-based fracture propagation in solid mechanics. The phase-field model is based on a thermodynamically-consistent version proposed by Miehe, Welschinger, Hofacker. The main focus is on goal-oriented functional evaluations using a partition-of-unity dual-weighted residual estimator for accurate measurement of, for example, point values, stresses, or crack opening displacements. Our developments are substantiated with a numerical test.

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1 Introduction

The purpose of this study is on a posteriori error analysis accompanied with local mesh adaptivity for quasi-static phase-field fracture propagation problems. Specifically, we concentrate on goal-oriented error estimation with the dual-weighted residual (DWR) method [1]. Here, a novel variational-based DWR localization technique is applied [2] that uses a partition-of-unity and avoids evaluation of strong forms. A variational framework for brittle fracture was first proposed by Francfort and Marigo [3] (supplemented with numerical simulations in [4]) and later modified by Miehe et al. [5] in order to obtain a thermodynamically-consistent phase-field model for brittle fracture. Numerical realization of such variational techniques for fracture are based on Ambrosio-Tortorelli approximations in which discontinuities in the displacement field across the lower-dimensional crack surface are approximated by an auxiliary function φ. The latter one can be viewed as an indicator function, which introduces a diffusive transition zone between the broken and the unbroken material. This zone has half bandwidth ε, the so-called model regularization parameter. Specifically, the resulting problem is a variational inequality because of a fracture irreversibility constraint. Consequently, while combining phase-field fracture with the DWR method, we borrow ideas from Rannacher and Suttmeier [6] who formulated DWR techniques for elasto-plasticity with similiar challenges.

2 The forward problem

We are interested in the following system: Let $V := H_0^1(\Omega)$ and
$W_{in} := \{w \in H^1(\Omega) |\, w \leq \varphi^{n-1} \leq 1$ a.e. on $\Omega; \varphi^{n-1}$ being the previous time step solution$\}$ be the function spaces we work with here; and for later purposes we also need $W := H^1(\Omega)$. For simplicity we consider in this entire study scalar-valued displacements (i.e., a modified version of Poisson's problem as displacement equation).

Formulation 1 (Euler-Lagrange system of phase-field fracture propagation) Find scalar-valued displacements and a scalar-valued phase-field variable, i.e., $(u, \varphi) \in V \times W$ such that

$$\Big(\big((1-\kappa)\varphi^2 + \kappa\big)\, \sigma(u), e(w) \Big) = 0 \quad \forall w \in V, \tag{1}$$

and

$$(1-\kappa)(\varphi\, \sigma(u) : e(u), \psi - \varphi) + G_c \Big(-\frac{1}{\varepsilon}(1-\varphi, \psi-\varphi) + \varepsilon(\nabla\varphi, \nabla(\psi-\varphi)) \Big) \geq 0 \quad \forall\, \psi \in W_{in} \cap L^\infty(\Omega), \tag{2}$$

and the crack irreversiblity constraint $\partial_t \varphi \leq 0$.

In Formulation 1, κ is a positive regularization parameter for the elastic energy, with $\kappa \ll \varepsilon$, and G_c is the critical energy release rate. Linear elasticity with the standard stress-strain relationship is defined as $\sigma := \sigma(u) = 2\mu e(u) + \lambda \operatorname{tr}(e(u))I$. Here, μ and λ are material parameters, $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the strain tensor, and I the identity matrix. Here, the variational inequality is treated with penalization (a discussion as well as a suggestion of a more sophisticated scheme are provided in [7]).

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3 Goal-oriented error estimation with the dual-weighted residual method

Let J be the goal (or target) functional. A relevant example is a point value evalation, i.e., $J(U) = U(x_0, y_0)$ of the function u in the point (x_0, y_0). Solving a dual problem [1], the a posteriori error estimator to such a functional reads [1]:

$$|J(U) - J(U_h)| \leq \sum_{T \in \mathbb{T}_h} \rho_T(U_h)\omega_T(Z),$$

with the local residuals $\rho_T(U_h)$ and sensitivity weights $\omega_T(Z)$. Here, h denotes as usually the spatial discretization parameter. The dual solution $Z \in V$ cannot be determined analytically but must be solved numerically as the primal problem, i.e, we search for $Z_h \in V_h$. In the following, we use a recently introduced variational localization formulation [2] that only needs a partition-of-unity (PU), $\sum_i \chi_i \equiv 1$ rather than partial integration to obtain the strong operator and face terms. Specifically as PU, we consider the space of piece-wise bilinear elements $V_h^{(1)}$ (without restrictions on Dirichlet boundaries) with usual nodal basis $\{\chi_h^i,\ i = 1, \ldots, N\}$. Taking into account that the phase-field variable is an auxiliary variable φ that helps to determine the crack path, we formulate the error estimator 'only' in terms of the physical displacements. Furthermore, we restrict ourselves to study estimates for varying h while keeping the regularization parameters ε and κ fixed.

Proposition 1 We have the (reduced) a posteriori error estimate for the displacement-phase-field problem:

$$J(u_\varepsilon) - J(u_{\varepsilon,h}) = \sum_i^N \eta_i = \sum_i^N \Big(-\big((1-\kappa)\tilde{\varphi}^2 - \kappa\big)\ \sigma(u), e(w) \Big),$$

where η_i are the local (nodal-based) error indicators and the *weighting function* (composed of the dual problem) [1] is defined as $w := (w_{2h}^{(2)} - w_h)\chi_h^i$. Furthermore, $\tilde{\varphi}$ is an extrapolation of the true φ as suggested and discussed in [8].

4 A numerical example

In this final section, a numerical example substantiates our developments. We consider the slit domain in $\Omega := (-1, 1)^2$ with a displacement discontinuity (i.e., the crack). In [9, 10], a manufactured solution including boundary conditions for this displacement field has been constructed (see the left subfigure in Figure 1 for a visualization). The regularization parameter ε is fixed by $h_{coarse} = 8.84e - 2$. The other model and material parameters are given as: $\kappa = 10^{-14}$, $G_c = \lambda_{G_c}^2 \times \sqrt{\pi/2}$, $\lambda_{G_c} = 1.0$, $\mu = 1.0$. The goal functional is defined as $J(u_\varepsilon) = u_\varepsilon(-0.75, -0.75)$ and the error $J(u_\varepsilon) - J(u_{\varepsilon,h})$ (for fixed ε) is subject of our investigation. A related example has been considered in [11]. The primal problem is computed with Q_1^c (continuous bilinears) finite elements for both the displacement as well as the phase-field approximation.

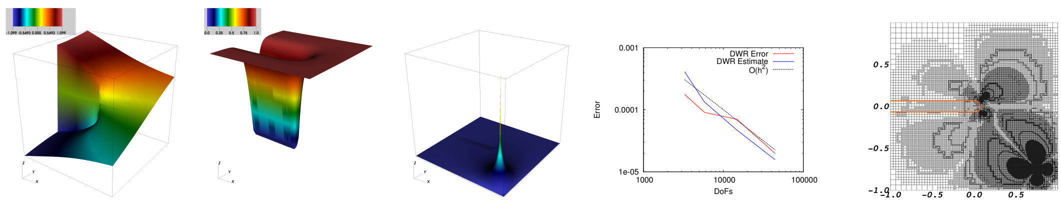

Fig. 1: Numerical example: phase-field fracture approximation in the slit domain. In the left sub-figure the displacement field is shown in a three-dimensional view in order to highlight the displacement discontinuity. Then, the phase-field function is observed; with values 0 in the fracture and 1 outside and smooth interpolation in between. In the third figure, the dual functional, here a Dirac functional, represents a point evaluation. Next, the error and the DWR estimate are displayed including a comparison of the convergence order. In the right sub-figure, the resulting locally adapted mesh and the crack contour $\varphi = 0.1$ (colorized in orange) are shown. Observing the error and the DWR estimate yield a relative good effectivity index while both show similar convergence order.

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