Approximation theory in FEM for interpolation and numerical integration

for elliptic PDEs of order 2

THOMAS WICK

Friday, April 11, 2008

Wissenschaftliches Rechnen
Fachbereich 6 - Mathematik
UNIVERSITÄT SIEGEN

Contents

1 Problem and Sobolev spaces 7

2 Approximate solution of A and b 9

3 Interpolation with exact polynomial integration 11

4 Integration with numerical quadrature 13

5 Summary and examples 23

Glossary of Symbols

Ω	bounded, convex, polygonal domain in $\mathbb{R}^n$
Ω°	interior points of Ω
$P_k(\Omega)$	set of polynomials with degree k on $\Omega \subset \mathbb{R}^n$
S_h^m	$= V_h$, space of finite element functions with degree $\leq m$
$C^l(\Omega)$	space of l-times continuously differentiable functions, with $l \in \mathbb{N}$
$C^\infty(\Omega)$	space of infinite continuously differentiable functions
$C_0^l(\Omega)$	space of l-times continuously differentiable functions with compact domain
$L^p(\Omega)$	$p \in ([1, \infty))$, space of real- or complex-valued functions such that the Lebesgue integral of $\|f\|^p$ over Ω exists and is finite
$H^{r,p}(\Omega)$	$=\{v \in L^p(\Omega) : \partial^\alpha v \in L^p(\Omega)$ for $\|\alpha\| \leq r,\ p \in [1, \infty)\}$
$H^r(\Omega)$	$(p = 2)$: Sobolev space of L^2-functions which are square-integrable with order to r
$H_0^r(\Omega)$	subspace of $H^r(\Omega)$ of functions vanishing on $\partial\Omega$

$a(u, v)$	bilinear form of a PDE
$\tilde{a}(\tilde{u}, v)$	approximated bilinear form for a quadrature formula
$l(v)$	$= (f, v)_0$, linear functional
$\tilde{l}(v)$	approximates linear functional
u	exact solution of a PDE
u_h	discretizated solution by Ritz-Galerkin procedure
$\tilde{u}_h$	approximated solution by numerical integration

$(\cdot, \cdot)_0$	inner product on $L^2(\Omega)$
$\|\| \cdot \|\|_0$	L^2-norm
$(\cdot, \cdot)_{r,T}$	inner product on $H^r(T)$. If $T = \Omega$ then $(\cdot, \cdot)_r$
$\|u\|_{r,p,T}$	$= \left(\sum_{\|\alpha\|=r} \int_a^b \|\partial^\alpha u\|^p \right)^{\frac{1}{p}}$, semi-norm on $H^{r,p}(T)$. We often use $\|u\|_{r,p}$ if $T = \Omega$
$\|\|u\|\|_{r,p,T}$	$= \left(\sum_{\|\alpha\|\leq r} \int_a^b \|\partial^\alpha u\|^p \right)^{\frac{1}{p}}$, norm on $H^{r,p}(T)$

α	$= (\alpha_1, \ldots, \alpha_n)$, multi-index
$\|\alpha\|$	$= \sum_{i=1}^n \alpha_i$, sum of α
∂_l	partial derivation $\frac{\partial}{\partial_{x_l}}$
∂^α	partial derivation of order α
∇	$:= (\partial_1, \ldots, \partial_n)^T$ Nabla-operator

1 Problem and Sobolev spaces

This report covers important numerical aspects of the finite element method: The order of convergence for an approximated solution by numerical integration. That means, the rapidity of convergence for a sequence of approached solutions to the exact solution. This aspect has consequences in the implementation of finite element algorithms.

In the finite element course we learned something about the error of an approximate solution u_h (see Céa-Lemma). Now, a second error estimate for $\tilde{u}_h$ becomes important which is a consequence of numerical quadrature, we denote it *error of consistence*, named *first Lemma from Strang*. Our focus is combining both errors and give some answers about the order of convergence in the H_1-norm $||\cdot||_1$, that means

$$||u - \tilde{u}_h||_1 \leq \underbrace{||u - u_h||_1}_{\text{Céa}} + \underbrace{||u_h - \tilde{u}_h||_1}_{\text{Strang}}$$

This investigation leads to practical results in solving integrals with numerical integration. One could say which degree of quadrature formula is necessary integrating finite elements at a given degree.

This discussion covers two possible ways: First, we study interpolation by Lagrange with exact integration. The second way illustrates approximation theory for numerical quadrature formulae, the most common way solving integrals in FEM.

Therefore, we study an elliptic variational problem in two dimensions. This includes the model problem (Poisson problem). The results also hold for elliptic PDE's of second order in 1D.

Problem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded, convex, polygonal domain. The task is finding a weak solution $u \in H_0^1(\Omega)$ for an elliptic boundary value problem in $\mathbb{R}^2$ with homogeneous Dirichlet conditions*

$$u \in H_0^1(\Omega): \quad a(u,v) = (f,v)_0 \quad \textit{for } v \in H_0^1(\Omega)$$

with

$$a(u,v) = \sum_{i,k=1}^{2} \int_\Omega a_{ik} \partial_k u \partial_i v \, dx \tag{1}$$

and coefficient functions $a_{ik} := a_{ik}(x)$.

We assume a regular triangulation $\mathbb{T}_h = \{T\}$ from $\overline{\Omega}$ with properties

i) $\overline{\Omega} = \bigcup_i T_i$

ii) $T_i^\circ \cap T_\nu^\circ = \varnothing, \quad \nu \neq i$

For more information see [4] p. 58.

Introduction of Sobolev spaces

The Sobolev spaces are Hilbert spaces based on the concept of weak derivatives. By $L^2(\Omega)$ we denote the space of measurable real-valued functions defined on the domain

Ω that are square-integrable in the sense of Lebesgue. The space $L^2(\Omega)$ with respect to the inner product

$$(u,v)_{L^2} = (u,v)_0 := \int_\Omega u(x)\,v(x)\,dx, \quad u,v \in L^2(\Omega)$$

is a Hilbert space. This inner product forms the norm

$$||u||_0 = \sqrt{(u,u)_0}\,.$$

Definition 1.2 *(Weak derivation)*
A function $u \in L^2(\Omega)$ is said to have a weak derivative $v = \partial^\alpha u$ in $L^2(\Omega)$ if $v \in L^2(\Omega)$ and

$$(\phi, v)_0 = (-1)^{|\alpha|}(\partial^\alpha \phi, u)_0 \quad \phi \in C_0^\infty(\Omega)\,.$$

Definition 1.3 *(Sobolev spaces)*
Let $m \in \mathbb{N}_0$. All functions $u \in L^2(\Omega)$ with weak derivatives ∂^α for all $|\alpha| \leq r$ belong to $H^r(\Omega)$. The inner product is defined by

$$(u,v)_r := \sum_{|\alpha|\leq r} (\partial^\alpha u, \partial^\alpha v)_0$$

This inner product generates the (semi-)norms

$$|u|_r := \sqrt{\sum_{|\alpha|=r} ||\partial^\alpha u||_0^2}$$

and the norm $$||u||_r := \sqrt{\sum_{|\alpha|\leq r} ||\partial^\alpha u||_0^2}$$

The space $H^r(\Omega)$ equipped with $||\cdot||_r$ is a Hilbert space. This seminar only needs $H_0^1(\Omega)$.

The finite element space S_h^m contains polynomial functions and is defined by

$$S_h^m = \{v \in H_0^1(\Omega) : v|_T \in P_m(T)\}$$

Ritz-Galerkin procedure

An approximate solution $u_h \in S_h^m$ for $u \in V$ is determined by the following idea. The variational formulation delivers a solution $u_h \in S_h^m$ if

$$a(u_h, v_h) = (f, v_h)_0 \quad \forall v_h \in S_h^m$$

The finite dimensional space S_h^m approximates V. This one satisfies a basis, for example

$$S_h^m =< w_1, \ldots, w_N >, \quad \dim(S_h^m) = N$$

The solution vector is given by a linear combination

$$u_h = \sum_{j=1}^N \alpha_j w_j$$

The determination of the discrete solution u_h is equivalent to the solution of the linear equation system

$$Au = b$$

with

$$A = (a(w_i, w_j))_{ji} \in \mathbb{R}^{N\times N}, \quad u = (\alpha_1, \dots, \alpha_N)^T \in \mathbb{R}^N, \quad b = (b_1, \dots, b_N)^T, b_j = (f, w_j)$$

Since we have an elliptic problem, i.e the bi-linear form a is elliptic, the matrix A is symmetric and positive definite. It follows immediately that the ellipticity of the approximate bi-linear form $\tilde{a}$ is of great importance.

The basis of all error estimates in FEM is the so-called Céa-Lemma which we explain as follows.

Theorem 1.4 *(Céa-Lemma)*
Let the bi-linear form a be V-elliptic in $H_0^1(\Omega)$. Furthermore u and u_h are solutions of the variational problem in V respectively $S_h = V_h \subset V$, with dim$(S_h) < \infty$. It holds

$$||u - u_h||_1 \leq \frac{C}{\alpha} \inf_{v_h \in S_h} ||u - v_h||_1$$

Proof. See [4] p. 53

The aspect of convergence of u_h to the exact solution u is directly determined by the function spaces S_h, which approximate u. The fundamental question discusses the degree of accuracy of the elements of the system matrix

$$A = (a(w_k, w_j))_{j,k}$$

and the right-hand-side

$$b = ((f, w_j))_j$$

for optimal convergence.

2 Approximate solution of A and b

There is often no chance or an enormous computational cost solving matrix A and right-hand side b. In principle we have the reasons

i) The first case puts $a_{ij} = \delta_{ij}$ in (1), so the computation uses integration formulae for polynomials of degree $2m - 2$. Exact formulae for higher m need many computations but convergence is sure for quadrature rules of lower order.

ii) In general, the coefficient functions a_{ij} are not constant. Therefore, exact computations for the elements of A_{ij} and b_i are not possible.

iii) Numerical integration is the favorite way to determine isoparametric elements.

The next ideas where introduced by Strang and generalize the Céa-Lemma. Already known is the error of approximation and now a second estimate becomes important, named *error of consistence*.

Calculations will be done on a reference element subsequently affine-transformed on each cell (triangle).

Idea, deriving an approximate solution

The matrix $A = (A_{ij})_{i,j=1}^{N}$ and right-hand-side $b = (b_i), i = 1, \dots, N$ leads to the linear equation system

$$Au = b$$

Instead of solving this one, an approximate linear equation system is given by

$$\tilde{A}\tilde{u} = \tilde{b} \tag{2}$$

with $\tilde{A} = (\tilde{A}_{ij})_{i,j=1}^{N}$ and $\tilde{b} = (\tilde{b}_i), i = 1, \dots, N$. The solution of (2) brings us the approximate solution vector

$$\tilde{u}_h = \sum_{i=1}^{N} \tilde{\alpha}_i w_i \in S_h^m .$$

Now the error

$$e_h = u_h - \tilde{u}_h$$

arises. Next we give an idea for the corresponding variational formulation of (2). We work with two linear combinations of S_h^m:

$$v_h = \sum_{i=1}^{N} \xi_i w_i \quad \text{and} \quad w_h = \sum_{j=1}^{N} \eta_j w_j$$

These are necessary for the approximate bilinear form $\tilde{a}(v_h, w_h)$ and the approximate right-hand-side $\tilde{l}(v_h)$:

$$\tilde{a}(v_h, w_h) = \sum_{i,j=1}^{N} \tilde{A}_{ij} \xi_i \eta_j \quad \text{and} \quad \tilde{l}(v_h) = \sum_{i=1}^{N} \tilde{b}_i \xi_i \tag{3}$$

This leads to the variational formulation $(\tilde{V})$

$$\tilde{u}_h \in S_h^m : \quad \tilde{a}(\tilde{u}_h, v_h) = \tilde{l}(v_h) \quad \forall v_h \in S_h^m \qquad \Leftrightarrow \qquad \tilde{A}\tilde{u} = \tilde{b}$$

The following lemma describes an error estimate for $u_h - \tilde{u}_h$.

Lemma 2.1 *(first Lemma from Strang)*
The approximate bilinear form $\tilde{a}$ and the linear functional $\tilde{l}$ satisfying the condition

$$||v_h||_1^2 \leq \gamma \tilde{a}(v_h) \qquad \textit{(Uniform ellipticity of } \tilde{a}\textit{)} \tag{4}$$

uniformly in $0 < h \leq h_0$ and also holds the estimate

$$|(a - \tilde{a})(u_h, v_h)| + |(f, v_h)_0 - \tilde{l}(v_h)| \leq ch^\tau \cdot ||v_h||_1 , \quad v_h \in S_h^m \tag{5}$$

Then the problems $(\tilde{V})$ have unique solutions $\tilde{u} \in S_h^m$. The quantitative error estimate of the Ritzapproximation $u_h \in S_h^m$ is given by

$$||u_h - \tilde{u}_h||_1 \leq c\gamma h^\tau$$

Proof.
The unique solvability may be proofed with the Theorem of Lax-Milgram. See [4] p. 37 respectively [3] p.41. We derive the quantitative error estimation by the following calculation. Denote $e_h = v_h = u_h - \tilde{u}_h$,

$$\begin{aligned}\tilde{a}(e_h) &= \tilde{a}(e_h, e_h) = \tilde{a}(u_h - \tilde{u}_h, e_h) = \tilde{a}(u_h, e_h) - \tilde{a}(\tilde{u}_h, e_h) \\ &= \tilde{a}(u_h, e_h) - a(u_h, e_h) + a(u_h, e_h) - \tilde{a}(\tilde{u}_h, e_h) \\ &= (\tilde{a} - a)(u_h, e_h) + a(u_h, e_h) - \tilde{a}(\tilde{u}_h, e_h) \\ &= (\tilde{a} - a)(u_h, e_h) + (f, e_h)_0 - \tilde{l}(e_h) \\ &\leq ch^\tau ||e_h||_1\end{aligned}$$

The last estimation follows from second assumption (5). Since we have (4), we can conclude:

$$||e_h||_1^2 \leq \gamma \tilde{a}(e_h) \leq \gamma c h^\tau ||e_h||_1$$

Division by $||e_h||_1$ shows the assertion.

□

With the aid of (3) we have a corresponding formulation for uniform ellipticity of $\tilde{a}$:

$$\tilde{a}(v_h) = \sum_{i,j} \tilde{A}_{ij} \xi_i \xi_j \geq \gamma_1 |\xi|^2 \quad \forall \xi \in \mathbb{R}^n,\ \gamma_1 > 0 \tag{6}$$

Hence the coefficient matrix is uniformly definite in the variable x.

We point out two different ways for an approximate determination of A and b. In this context, we assume that the coefficient functions a_{ij} and the power vector f are sufficiently smooth,

$$a_{ij}, f \in L^\infty(E), \quad i, j = 1, 2$$

Secondly, we work with an *affine family* of finite elements, so we have the polynomial space $P_m(E)$ for the reference element. Interpolation and quadrature formulae were derived on that reference element and later transformed on each cell of Ω.

3 Interpolation with exact polynomial integration

There is a triangulation $\mathbb{T}_h$. Each triangle $T \in \mathbb{T}_h$ satisfies polynomial interpolation for a_{ij} and f with

$$\tilde{a}_{ij}, \tilde{f} \in P_{r-1}(T), \quad i, j = 1, 2$$

The error is determined by the remainder and we find the uniform estimation

$$||a_{ij} - \tilde{a}_{ij}||_\infty + ||f - \tilde{f}||_\infty = \mathcal{O}(h^r) \tag{7}$$

This error is independent from step width h and holds for functions $a_{ij}, f \in C^r(T)$. Higher differentiability leads not to a better error estimation. See [5] p. 194ff.

We compute the elements of $\tilde{A}$ and right-hand-side $\tilde{b}$ with

$$\tilde{A}_{ij} = \int_\Omega \sum_{\nu,\mu=1}^{2} \tilde{a}_{\nu\mu} \partial_\mu w_i \, \partial_\mu w_j \, dx, \quad i,j = 1, \ldots, N,$$

$$\text{and} \quad \tilde{b}_i = \int_\Omega \tilde{f} w_i \, dx, \quad i = 1, \ldots, N.$$

The advantage of these elements is exact integration. Therefore we have to evaluate the following integrals,

$$\int_T x^\beta \, dx, \quad 0 \leq |\beta| \leq 2m + r - 3$$

Calculation.
We give an answer that the degree of the polynomials must be $0 \leq |\beta| \leq 2m + r - 3$. The assumption says $w_i \in S_h^m$ which means $w_i|_T \in P_m(T)$. The coefficient function $\tilde{a}_{ij}$ is a polynomial of $\in P_{r-1}(T)$. Then

$$\tilde{a}_{\nu,\mu} \partial_\nu w_i \, \partial_\mu w_j \in P_{(r-1)+2m-2}(T)$$

The proof is finished.

□

An error estimation is given by Lemma (2.1). First task is to check the assumptions. Consider $v_h, w_h \in S_h^m$:

$$\begin{aligned}
|(a - \tilde{a})(v_h, w_h)| &= |a(v_h, w_h) - \tilde{a}(v_h, w_h)| \\
&= \left| \int_G \sum_{\nu,\mu=1}^{2} a_{\nu\mu} \partial_\mu v_h \, \partial_\nu w_h \, dx - \int_G \sum_{\nu,\mu=1}^{2} \tilde{a}_{\nu\mu} \partial_\mu v_h \, \partial_\nu w_h \, dx \right| \\
&= \left| \int_G \sum_{\nu,\mu=1}^{2} (a_{\nu\mu} - \tilde{a}_{\nu\mu}) \partial_\mu v_h \, \partial_\nu w_h \, dx \right| \\
&\leq c||a - \tilde{a}||_\infty \cdot \left| \int_G \sum_{\nu,\mu=1}^{2} \partial_\mu v_h \, \partial_\nu w_h \, dx \right| \\
&\overset{\text{C.S.}}{\leq} c||a - \tilde{a}||_\infty \cdot ||v_h||_1 \cdot ||w_h||_1 \\
&\overset{(7)}{\leq} ch^r ||v_h||_1 \cdot ||w_h||_1
\end{aligned} \tag{8}$$

We proof the second part

$$\begin{aligned}
|(f, v_h) - \tilde{l}(v_h)| &= \left| \int_G f \, v_h \, dx - \int_G \tilde{f} \, v_h \, dx \right| \\
&= \left| \int_G (f - \tilde{f}) \, v_h \, dx \right| \\
&\leq c||f - \tilde{f}||_\infty \cdot ||v_h||_\infty \\
&\overset{(7)}{\leq} ch^r \cdot ||v_h||_1
\end{aligned}$$

We complement the proof showing (4),

$$\begin{aligned}
||v_h||_1^2 &\le c\,a(v_h) && \text{(V-ellipticity)}\\
&\le c|a(v_h) - \tilde{a}(v_h) + \tilde{a}(v_h)| && \text{(expansion with } \tilde{a}(v_h)\text{)}\\
&\le c|a(v_h) - \tilde{a}(v_h)| + c|\tilde{a}(v_h)| && \text{(triangle inequality)}\\
&= c|(a-\tilde{a})(v_h)| + c|\tilde{a}(v_h)|\\
&\le ch^r||v_h||_1^2 + c\,\tilde{a}(v_h) && \text{(set } v_h = w_h \text{ in (8))}
\end{aligned}$$

The last estimate is a result from $\tilde{a}(v_h) > 0$. For sufficient small step size $0 < h \le h_0$ we can conclude

$$||v_h||_1^2 \le c\,\tilde{a}(v_h)$$

Now, Lemma (2.1) delivers the error of the approximate Ritzapproximation

$$||u_h - \tilde{u}_h||_1 \le ch^r$$

Total Error

We refrain the roles of the different solutions

- u : original function which is to approximate,
- the function u is approached by the solution u_h of the Ritz-procedure,
- for lower computational cost, u_h is approximated itself by $\tilde{u}_h$.

The triangle inequality shows

$$||u - \tilde{u}_h||_1 \le ||u - u_h||_1 + ||u_h - \tilde{u}_h||_1 = \mathcal{O}(h^m + h^r)$$

The order for optimal convergence is $r \ge m$. The error of the interpolating functions can be neglected in comparison with the procedure-error if $r > m$.

4 Integration with numerical quadrature

In this context numerical integration formulae based on interpolated functions. The most common formulae is

$$f_T(g) = \int_T g\,dx \sim Q_T(g) = \sum_{i=1}^{L} \omega_i g(\xi_i) \tag{9}$$

with distinct quadrature points $\xi_i \in T$ and quadrature weights ω_i. For construction of these formulae, please visit other literature.

Affine transformation rules

We define the unit triangle E as the reference element. Let $\sigma_T : E \to T$ be affine-linear mapping of E onto the triangle T. Then we have the properties

$$x = \sigma_T(\hat{x}) = B_T\hat{x} + b_T, \quad x \in T, \hat{x} \in E$$

with the functional matrix B_T.

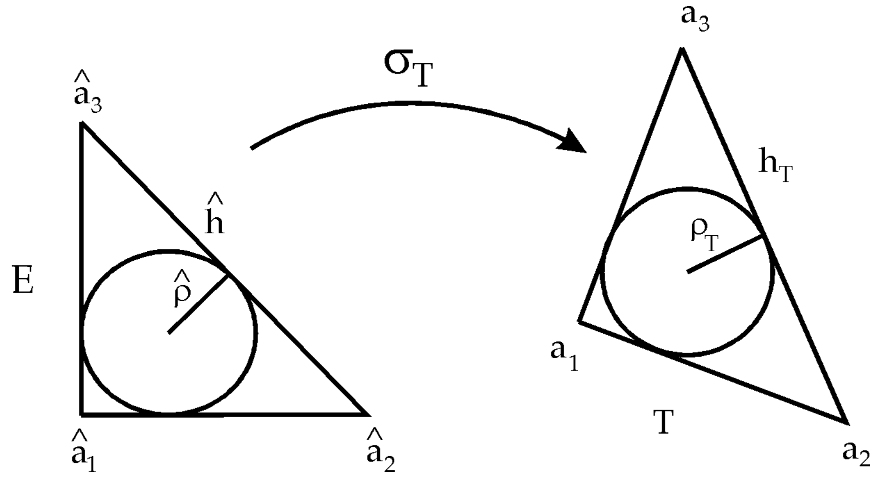

Figure 1: Reference element E and arbitrary cell (triagle) T

Example.
Let $E = \text{conv}\{\hat{a}_1, \hat{a}_2, \hat{a}_3\}$ with $\hat{a}_1 = (0,0)^T, \hat{a}_2 = (1,0)^T, \hat{a}_3 = (0,1)^T$. The triangle T has the coordinates $a_1 = (3, 1/4), a_2 = (4, -1/2), a_3 = (17/4, 1)$.

A bijective mapping between E and a triangle T is given by $\sigma_T : E \to T$ with $\sigma_T(\hat{x}) = B_T\hat{x} + b$, where $b = a_1$. The functional matrix is defined by

$$B_T = (a_2 - a_1, a_3 - a_1) = \begin{pmatrix} 1 & \frac{5}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix}$$

The determinant of B_T is of order h^2. That means $\det B_T = \mathcal{O}(h^2)$.

We obtain the relation, also known as "affine-equivalence" between two Lagrange elements,

$$\hat{v}(\hat{x}) := v(x), \quad \hat{x} = \sigma_T^{-1}x, \, x \in T$$

between functions on E and T. The polynomial space for the reference element is charakterized by $P_m(E)$. A (linear) quadrature rule on E satisfies

$$Q_E(g) = \sum_{i=1}^{L} \hat{\omega}_i g(\hat{\xi}_i), \quad \hat{\omega}_i := \int_E \hat{L}_i(\hat{x}) d\hat{x}, \, \hat{L}_i(\hat{x}) = \prod_{k=0}^{n} \frac{\hat{x} - x_k}{x_i - x_k}$$

with quadrature points $\hat{\xi}_i \in E$ and positive weights $\hat{\omega}_i$, $i = 1, \ldots, L$.

Affine transformation of QF on E gives quadrature rules on each element T with

$$Q_T(g) = \sum_{i=1}^{L} \omega_i g(\xi_i)$$

with quadrature points $\xi_i = \sigma_T \hat{\xi}_i$ and weights $\omega_i = |\text{det} B_T|\hat{\omega}_i, i = 1, \dots, L$. Then

$$Q_T(g) := \sum_{i=1}^{L} \omega_i g(\xi_i) = \sum_{i=1}^{L} |\text{det} B_T|\hat{\omega}_i \hat{g}(\hat{\xi}_i) = |\text{det} B_T|\, |Q_E(\hat{g})|$$

There is a short introduction of the very important transformation rule for integrals. It follows from the substitution rule in 1D.

Lemma 4.1 *Let T_1 and T_2 open subsets of $\mathbb{R}^n$ and $\sigma : T_1 \to T_2$. Then, and only then, the function f on T_2 is integrable if $(f \circ \sigma) \cdot |det\sigma'|$ is integrable over T_1. It is*

$$\int_{T_1} f(\sigma(x)) \cdot |det\sigma'|\, dx = \int_{T_2} f(y)\, dy$$

Proof. See [7] p. 299. It follows directly for an affine mapping $\sigma\hat{x} = B\hat{x} + b$, with derivation $D(\sigma\hat{x}) = D(B\hat{x} + b) = B$, that

$$\int_T v(x)\, dx = |\text{det} B_T| \int_E \hat{v}(\hat{x})\, d\hat{x}$$

The next two estimations can directly derived from the geometry of our triangulation. Remember the use of triangles in a regular triangulation. "Regular" means, that

$$\varrho_T \geq c h_T \quad \forall h, T, \, c > 0$$

where h_T charakterises the maximum diameter for any triangle, that means the longest side. In general we have the following two situations

$$||B_T||_2 \leq \frac{h_T}{\hat{\varrho}} = c_1 h_T \quad \text{and} \quad ||B_T^{-1}||_2 \leq \frac{\hat{h}}{\varrho_T} = c_0 \varrho_T^{-1} \tag{10}$$

The symbol $||B||_2$ means

$$||B||_2 = \left(\sum_{i,k} |b_{ik}|^2 \right)^{\frac{1}{2}}$$

Furthermore there is an estimation for determinants as follows

$$||B_T^{-1}||_2^{-n} \leq |\text{det} B_T| \leq ||B_T||_2^n$$

which may be proofed with the Laplace expansion theorem. Then

$$c_0 \varrho_T^n \leq |\text{det} B_T| \leq c_1 h_T^n$$

and

$$||B_T||_2\, ||B_T^{-1}||_2 \leq C \frac{h}{\varrho}$$

It follows in 2D ($n = 2$)

$$C_0 h_T^2 \leq \frac{1}{||B_T^{-1}||_2^2} \leq |\det(B_T)| \leq ||B_T||_2^2 \leq C_1 h_T^2 \tag{11}$$

At last we mention

$$c_0|\widehat{\nabla}\hat{v}(\hat{x})| \leq h_T|\nabla v(x)| \leq c_1|\widehat{\nabla}\hat{v}(\hat{x})|, \quad x \in T, \hat{x} = \sigma_T^{-1} x \tag{12}$$

Calculation.
Estimation (12) follows from the compound rule of three

$$\widehat{\nabla}\hat{v}(\hat{x}) = B_T^T \nabla v(x) \tag{13}$$

We have

$$|\nabla v(x)| = |\nabla v(\sigma(\hat{x}))| = |B^{-T}\widehat{\nabla}\hat{v}(\hat{x})| \leq ||B^{-T}||_2|\widehat{\nabla}\hat{v}(\hat{x})|$$

and on the other hand

$$|\widehat{\nabla}\hat{v}(\hat{x})| = |B^T \nabla v(x)| \leq ||B^T||_2 |\nabla v(x)|$$

Working with (10) show (12).

Numerical quadrature

We set QF as an abbreviation for *quadrature formula*.

Definition 4.2 *(Order of a QF)*
A QF $Q_T(\cdot)$ on T is said to be of order r if it integrates exactly all polynomials of degree $r-1$,

$$Q_T(p) = \int_T p\, dx, \quad p \in P_{r-1}(T)$$

Another equivalent formulation is given by

$$Q_T(g) = \int_T P_T g\, dx, \quad P_T g \in P_{r-1}(T)$$

where P_T is an interpolating operator for g. Then, $Q_T(\cdot)$ is named as an interpolatary approximation of $f_T(\cdot)$.

The first result gives an estimation for the error of numerical quadrature. Therefore we need the seminorm

$$|u|_{r,1,T} = \sum_{|\alpha|=r} \int_T |\partial^\alpha u|\ dx$$

and we mention that $C^r(T)$ is dense in $H^{r,1}(T)$.

Lemma 4.3 *(Quantitative error of numerical quadrature)*
Assume $Q_T(g)$ is a QF of order $r \geq 3$ of the function $g \in H^{r,1}(T)$. Then

$$|f_T(g) - Q_T(g)| \leq ch^r |g|_{r,1,T} \quad \forall T \tag{14}$$

for constant c which is independent of T and h

Proof.
The requirement $r \geq 3$ will be proofed with the inequality from Sobolev which is a consequence of the embedding theorem. Note, that the case $r = 2$ is also possible but difficult to prove. The linear functional f_T is continuous with respect to the L^1-norm,

$$|f_T(g)| = \left| \int_T g\,dx \right| \leq \int_T |g|\,dx = |g|_{0,1,T}\,, \quad g \in L^1(T) \tag{15}$$

It follows for $g \in H^{r,1}(T)$ and all T

$$\begin{aligned} |f_T(g) - Q_T(g)| &= |f_T(g) - f_T(P_T g)| \qquad \text{(interpolated approximation of } f_T) \\ &= |f_T(g - P_T g)| \\ &\overset{(15)}{\leq} |g - P_T g|_{0,1} \\ &\leq ch^r |g|_{r,1} \qquad \text{(interpolation estimation)} \end{aligned}$$

We finished proof.

□

Definition 4.4 *(Admissibility of a QF)*
Each polynomial $q \in P_{m-1}(E)$, integrated by Q_E, has the property

$$\textit{For} \quad q(\hat{\xi}_i) = 0,\, i = 1, \ldots, L \quad \textit{follows} \quad q \equiv 0 \tag{16}$$

This property ensures that the Lagrange interpolation has a unique solution. The necessary condition for (16) requires that the number of interpolation points L is greater or equal to $\dim P_{m-1}$.

We need further results like a new norm, and after that definition, we give an estimation between QF and integral.

Definition 4.5 *(Norm on $P_{m-1}(E)/P_0(E)$)*
On the finite dimensional quotient space $P_{m-1}(E)/P_0(E)$ is a norm defined by

$$|||\hat{q}||| := \left(\sum_{i=1}^{L} \hat{\omega}_i \sum_{j=1}^{2} (\partial_j q)^2(\hat{\xi}_i) \right)^{\frac{1}{2}}$$

Proof.
We have to proof all attributes of a norm. But the friendly reader wants to show three properties. We only check,

DEFINITENESS.

Property (16) holds. Since the quadrature weights $\hat{\omega}_i$ are positive it follows for $q \in P_{m-1}(E)$,

$$\begin{aligned} &\sum_{i=1}^{L} \hat{\omega}_i \sum_{j=1}^{2} (\partial_j q)^2(\hat{\xi}_i) = 0 \\ \Rightarrow \quad &\partial_j q(\hat{\xi}_i) = 0, \quad 1 \leq j \leq 2, \quad 1 \leq i \leq L \end{aligned}$$

Remark that any $\partial_j q$ is an element of $P_{m-2}(E)$ because $q \in P_{m-1}(E)$. Since (16), $\partial_j q$ vanishes at $\hat{\xi}_i$. This implies $\partial_j q \equiv 0$, $j = 1, 2$ which shows the definiteness.

The reader may check

$$||q|| \geq 0, \quad ||\alpha q|| = |\alpha|\, ||q||, \, \alpha \in \mathbb{R}, \quad ||p + q|| \leq ||p|| + ||q||$$

□

Lemma 4.6 *(Ellipticity of QF)*
For some constant c which is independent of h we have

$$Q_T(|\nabla v|^2) \geq c \int_T |\nabla v|^2 dx \quad \forall v \in S_h^{m-1} \quad \forall T \tag{17}$$

Proof.
There is some work to do. First we have the norm of definition (4.5). A second norm on $P_{m-1}(E)/P_0(E)$ is given by $|\hat{q}|_{1,E}$. The finite dimension of $P_{m-1}(E)/P_0(E)$ leads to the equivalence of the two norms. It exists some constant $\hat{C}$, that

$$\hat{C}|\hat{q}|_1^2 \leq |||\hat{q}|||, \quad \hat{q} \in P_{m-1}(E)/P_0(E) \tag{18}$$

We compute

$$\begin{aligned}
Q_T(|\nabla p|^2) &\overset{(9)}{=} \sum_{i=1}^{L} \omega_i |\nabla p(\xi_i)|^2 \\
&= |\mathrm{det} B_T| \sum_{i=1}^{L} \hat{\omega}_i |\nabla p(\xi_i)|^2 \quad (\text{with } \omega_i = |\mathrm{det} B_T| \hat{\omega}_i) \\
&\overset{(12)}{\geq} c_0 |\mathrm{det} B_T| \frac{1}{h^2} \sum_{i=1}^{L} \hat{\omega}_i |\widehat{\nabla} \hat{q}(\hat{\xi}_i)|^2 \\
&\geq c_0' \cdot Q_E(|\widehat{\nabla} \hat{q}|^2) \\
&= c_0' |||\hat{q}|||, \quad \hat{q} \in P_{m-1}(E)/P_0(E)
\end{aligned}$$

We conclude with the aid of equivalence of the two norms (18)

$$\begin{aligned}
Q_T(|\nabla p|^2) &\geq c_0' |||\hat{q}||| \overset{(18)}{\geq} c_0' \hat{C} |\hat{q}|_{1,E}^2 \\
&= c_0 |\mathrm{det} B_T| h^{-2} |\hat{q}|_{1,E}^2 = c_0 |\mathrm{det} B_T| h^{-2} \int_E |\widehat{\nabla} \hat{q}|^2 \, d\hat{x} \\
&= c_0 |\mathrm{det} B_T| h^{-2} \int_E |B_T^T \nabla q|^2 \, d\hat{x} \\
&\geq c_0 h^{-2} \int_T ||B_T^T||^2 |\nabla q|^2 \, dx \\
&\overset{(11)}{\geq} c_0 h^{-2} C_1 h^2 \int_T |\nabla q|^2 \, dx \\
&= C \int_T |\nabla p|^2 \, dx
\end{aligned}$$

for $p \in S_h^{m-1}$. Which completes the proof. □

Error estimation

The central theorem leads to an error estimation of the FEM for numerical quadrature formulae.

We give the definitions of the approximate elements of the matrix $\tilde{A}_{ij}$ and the right-hand side $\tilde{b}_i$

$$\tilde{A}_{ij} = \sum_T Q_T \left(\sum_{\nu,\mu=1}^{2} a_{\mu\nu} \partial_\mu w_i \, \partial_\nu w_j \right), \quad i,j = 1,\dots,N$$

and

$$\tilde{b}_i = \sum_T Q_T(f w_i), \quad i = 1,\dots,N$$

The main Theorem is a consequence of the Lemma from Strang (2.1). First, we proof the conditions, then we have the error estimation for $u_h - \tilde{u}_h$.

Theorem 4.7 *Assume the quadrature formula, of order $r \geq 3$,*

$$Q_E(g) = \sum_{i=1}^{L} \hat{\omega}_i g(\hat{\xi}_i)$$

on E, which is admissible for $P_{m-1}(E)$. Let $r \geq m-1$, $m-2$. The degree of the finite elements is $m-1$ that is $u_h, v_h \in S_h^{m-1}$. The coefficient functions $a_{\nu\mu} \in L^\infty(E)$ are sufficiently smooth. Then we have

$$|(a - \tilde{a})(u_h, v_h)| \leq Ch^{r-m+2} ||v_h||_1, \quad v_h \in S_h^{m-1}$$

and

$$|(f, v_h)_0 - \tilde{l}(v_h)| \leq Ch^{r-m+2} ||v_h||_1, \quad v_h \in S_h^{m-1}$$

and also uniform ellipticity

$$c||v_h||_1^2 \leq \tilde{a}(v_h), \quad v_h \in S_h^{m-1}$$

The constants c, C depending on u.

Proof.
Set $m' := m-1$. Denote $v_h \in S_h^{m'}$:

$$\begin{aligned}
|(a - \tilde{a})(u_h, v_h)| &= |a(u_h, v_h) - \tilde{a}(u_h, v_h)| \\
&= \left| \sum_T \left[\int_T \sum_{\nu,\mu=1}^{2} a_{\nu\mu} \partial_\mu u_h \, \partial_\nu v_h \, dx - Q_T \left(\sum_{\nu,\mu=1}^{2} a_{\nu\mu} \partial_\mu u_h \, \partial_\nu v_h \right) \right] \right| \\
&= \left| \sum_T \sum_{\nu,\mu=1}^{2} \left[\int_T a_{\nu\mu} \partial_\mu u_h \, \partial_\nu v_h \, dx - Q_T \left(a_{\nu\mu} \partial_\mu u_h \, \partial_\nu v_h \right) \right] \right| \\
&\overset{\text{TRI}}{\leq} \sum_T \left| \sum_{\nu,\mu=1}^{2} \left[\int_T a_{\nu\mu} \partial_\mu u_h \, \partial_\nu v_h \, dx - Q_T \left(a_{\nu\mu} \partial_\mu u_h \, \partial_\nu v_h \right) \right] \right| \\
&\overset{(14)}{\leq} ch^r \sum_T \left| \sum_{\nu,\mu=1}^{2} a_{\nu\mu} \partial_\mu u_h \, \partial_\nu v_h \right|_{r,1,T}
\end{aligned}$$

The functions $a_{\nu\mu}$ are bounded with $||a_{\nu\mu}||_\infty \leq c$ which implies

$$
\begin{aligned}
|(a-\tilde{a})(u_h,v_h)| &\overset{(14)}{\leq} ch^r \sum_T \left| \sum_{\nu,\mu=1}^{2} a_{\nu\mu}\partial_\mu u_h \, \partial_\nu v_h \right|_{r,1,T} \\
&\overset{\text{C.S.}}{\leq} ch^r \sum_T \left(\sum_{\mu=1}^{2} |\partial_\mu u_h|^2_{r,1,T} \right)^{\frac{1}{2}} \cdot \left(\sum_{\nu=1}^{2} |\partial_\nu v_h|^2_{r,1,T} \right)^{\frac{1}{2}} \\
&\leq ch^r \sum_T ||u_h||_{r+1,T} \cdot ||v_h||_{r+1,T} \qquad (19)
\end{aligned}
$$

Now, a function $w_h \in S_h^{m'}$ on each triangle $T \in \mathbb{T}_h$ is a polynomial of $P_{m'}(T)$. Because of differentiability, we compute

$$||w_h||_{r+1,T} = ||w_h||_{m,T} \quad \text{for } r \geq m' \qquad (20)$$

The inverse estimation delivers

$$||w_h||_{m',T} \leq ch^{1-m'}||w_h||_{1,T} \qquad (21)$$

Using (20) and (21) for the function v_h show us

$$||v_h||_{r+1,T} \leq ch^{1-m'}||v_h||_{1,T} \qquad (22)$$

We obtain with (20) for u_h

$$||u_h||_{r+1,T} = ||u_h||_{m',T} \qquad (23)$$

We put the last two results and the Cauchy-Schwarz inequality in (19),

$$
\begin{aligned}
|(a-\tilde{a})(u_h,v_h)| &\leq ch^r \sum_T ||u_h||_{r+1,T} \cdot ||v_h||_{r+1,T} \\
&\overset{\text{C.S.}}{\leq} ch^r \left(\sum_T ||v_h||^2_{r+1,T} \right)^{\frac{1}{2}} \cdot \left(\sum_T ||u_h||^2_{r+1,T} \right)^{\frac{1}{2}} \\
&\overset{(22),(23)}{\leq} ch^{r-m'+1}||v_h||_1 \cdot \left(\sum_T ||u_h||^2_{m',T} \right)^{\frac{1}{2}} \\
&= ch^{r-m'+1}||v_h||_1 \cdot ||u_h||_{m'}
\end{aligned}
$$

We specify the last term with the Ritzapproximation u_h. Taking an interpolation operator $I_h u \in S_h^{m'}$ for u we conclude

$$
\begin{aligned}
||u_h||_{m',T} &= ||u_h - I_h u + I_h u - u + u||_{m',T} \\
&\leq ||u_h - I_h u||_{m',T} + ||I_h u - u||_{m',T} + ||u||_{m',T} \\
&\overset{(21)}{\leq} ch^{1-m'}||u_h - I_h u||_{1,T} + ||I_h u - u||_{m',T} + ||u||_{m',T} \\
&= ch^{1-m'}||u_h - u + u - I_h u||_{1,T} + ||I_h u - u||_{m',T} + ||u||_{m,T} \\
&\leq ch^{1-m'}||u_h - u||_{1,T} + ch^{1-m'}||u - I_h u||_{1,T} + ||I_h u - u||_{m',T} + ||u||_{m',T}
\end{aligned}
$$

Remember the result from approximation with I_h:

$$||u - I_h u||_{m'} \leq ch^{t-m'}|u|_t \leq ch^{t-m'}||u||_t \quad \text{for } u \in H^t(\Omega),\, 0 \leq m' \leq t.$$

Then, we have

$$\begin{aligned}||u_h||_{m',T} &\leq ch^{1-m'}||u_h - u||_{1,T} + c_1||u||_{m',T} + c_2||u||_{m',T} + ||u||_{m',T} \\ &\leq ch^{1-m'}||u_h - u||_{1,T} + C||u||_{m',T}\end{aligned}$$

The extension on Ω is

$$||u_h||_{m'} = \left(\sum_T ||u_h||^2_{m',T}\right)^{\frac{1}{2}} \leq ch^{1-m'}||u - u_h||_1 + c||u||_{m'}$$

Since $||u - u_h||_1 = \mathcal{O}(h^{m'})$, gives

$$\begin{aligned}|(a - \tilde{a})(u_h, v_h)| &\leq ch^{r-m'+1}||v_h||_1 \cdot ||u_h||_{m'} \\ &\leq ch^{r-m'+1}||v_h||_1 \left(ch^{1-m'}||u - u_h||_1 + c||u||_{m'}\right) \\ &= ch^{r-2m'+2}||u - u_h||_1||v_h||_1 + ch^{r-m'+1}||v_h||_1 \cdot ||u||_{m'} \\ &= ch^{r-m'+2}||v_h||_1 + c(u)h^{r-m'+1}||v_h||_1 \\ &\leq c(u)h^{r-m'+1}||v_h||_1\end{aligned}$$

We explain in words: The third line uses the order of convergence $||u - u_h|| = \mathcal{O}(h^{m'})$. The number $||u||_{m'}$ is not important for convergence, so it is some constant factor $c(u)$. The order of convergence of $h^{r-m'+1}$ is not as good as $h^{r-m'+2}$ which completes the last line. Remark that $r \geq 3$ and $r \geq m'$. Short summary of the result ($m' = m - 1$):

$$|(a - \tilde{a})(u_h, v_h)| \leq ch^{r-m+2}||v_h||_1, \quad c = c(u)$$

Analogous computation verifies

$$|(f, v_h)_0 - \tilde{l}(v_h)| \leq ch^{r-m+2}||v_h||_1$$

The last estimation shows uniform ellipticity of $\tilde{a}$. Let $v_h \in S_h^{m'}$ which implies $v_h|_T \in P_{m'}(T)$. Also, numerical stability requires positive weights and we refrain that the bilinear form a is elliptic. Then

$$\begin{aligned}\tilde{a}(v_h) &= \sum_T Q_T\left(\sum_{\nu,\mu=1}^{2} a_{\nu\mu}\partial_\mu v_h\, \partial_\nu v_h\right) \\ &\overset{(3),(6)}{\geq} c\sum_T Q_T(|\nabla v_h|^2) \qquad \text{(uniform ellipticity of } a\text{)} \\ &\overset{(17)}{\geq} c\sum_T \int_T |\nabla v_h|^2\, dx \\ &= c|v_h|_1^2 \\ &\geq c||v_h||_1^2 \qquad \text{(Poincaré for functions of } H_0^1(\Omega)\text{)}\end{aligned}$$

The last estimation follows from the equivalence of $||\cdot||_1$ and $|\cdot|_1$ on H_0^1.

The conditions of Lemma (2.1) are true and the proof is finished.

□

The previous theorem yields the order convergence of the FEM for numerical integration.

Corollary 4.8 *The conditions of Theorem* (4.7) *hold. Then, Lemma* (2.1) *leads to a quantitative error estimation, with* $\tau = r - m + 2$*,*

$$||u_h - \tilde{u}_h||_1 = \mathcal{O}(h^{r-m+2})$$

The total error is given by

$$||u - \tilde{u}_h||_1 \leq ||u - u_h||_1 + ||u_h - \tilde{u}_h||_1 = \mathcal{O}(h^{m-1} + h^{r-m+2})$$

Convergence is assured if $r \geq m - 1$*. In other words*

$$r' \geq m' \tag{24}$$

Remark that r' *denotes the degree of the quadrature rule and* m' *the degree of the finite elements.*

Examples.
For $r = m - 1$ we have,

$$||u - \tilde{u}_h||_1 = \mathcal{O}(h)$$

Optimal convergence is given if $r \geq 2m - 3$. In example $r = 2m - 3$:

$$||u - \tilde{u}_h||_1 = \mathcal{O}(h^{m-1})$$

The error of quadrature can be neglected if $r > 2m - 3$, because

$$||u - \tilde{u}_h||_1 = \mathcal{O}(h^{m-1})$$

is more important for the order of convergence.

Remark 4.9 *The inequality* (24) *is the main assertion of the report. But remind that* $r' >> 2m' - 1$ *(degree of quadrature rule very large in comparison to the optimal order of convergence) is not recommended because the calculation of the coefficient functions* $a_{ik}(x)$ *and the gradients* $\partial_\mu w_i\, \partial_\nu w_j, i, j = 1, \ldots N,\ \mu, \nu = 1, 2$ *could be very expensive for higher order quadrature rules.*

5 Summary and examples

Summary

This discussion sums the seminar. Central result is the Lemma of Strang (2.1) which gives a quantitative error estimation for convergence and leads to Theorem (4.7).

The main condition is the uniform ellipticity of the approximate bi linear form $\tilde{a}$.

If the conditions are not satisfied it may be possible that the linear equation system $Au = b$ is not solvable because matrix A is singular. We study one example for this situation.

Example 1

We discuss one simple example for a situation where the system matrix will be singular. Since $r' \geq m$ we have convergence of the procedure. Where r' means the degree of the quadrature rule and m is the degree of the FEM polynomials.

Ok! Let's start.

We assume the one-dimensional Laplace equation, take quadratic finite elements and use the middle-point rule as integration formula. So we solve

$$-u'' = f, \quad u(0) = (1) = 0$$

and take quadratic basic functions in the interval $[0,1]$ with equidistant step size $h = x_i - x_{i-1}$. There is an additional point required, so we take the middle point of the cells T_i,

$$x_{i-1/2} := \frac{1}{2}(x_{i-1} - x_i)$$

The construction of the quadratic function v satisfies

$$v(x) = \sum_{i=0}^{N} v_i \psi_i(x) + \sum_{i=1}^{N} v_{i-1/2} \psi_{i-1/2}(x)$$

with the following properties

i) $\psi_i(x_k) = \delta_{ik}, \quad \psi_i(x_{k-1/2}) = 0$

ii) $\psi_{i-1/2}(x_k) = 0, \quad \psi_{i-1/2}(x_{k-1/2}) = \delta_{ik}$

Now, we get three quadratic basic functions

$$\psi_i(x) = \begin{cases} \frac{(x-x_{i-1})(x-x_{i-1/2})}{(x_i-x_{i-1})(x_i-x_{i-1/2})}, & x \in T_i \\ \frac{(x-x_{i+1})(x-x_{i+1/2})}{(x_i-x_{i+1})(x_i-x_{i+1/2})}, & x \in T_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\psi_{i-1/2}(x) = \begin{cases} \frac{(x-x_{i-1})(x-x_i)}{(x_{i-1/2}-x_i)(x_{i-1/2}-x_{i+1})}, & x \in T_i \\ 0, & \text{otherwise} \end{cases}$$

Next task is constructing the derivatives and give the variational formulation of the one-dimensional Laplace problem. Then solving integrals with middle-point rule,

$$\int_a^b f(x)\,dx \approx (b-a)\,f\left(\frac{a+b}{2}\right) + \frac{1}{24}f''(\xi)(b-a)^3 \tag{25}$$

One know that the middle-point rule has degree 1. Using this quadrature formula for quadratic elements fails and the resulting systemmatrix A will have some zeros, so it is singular. For a local element we obtain

$$\begin{aligned}
\partial_x \psi_{i-1}(x) &= \frac{2}{h_i^2}(2x - x_{i-1/2} - x_i) \\
\partial_x \psi_{i-1/2}(x) &= \frac{2}{h_i^2}(x_i + x_{i-1} - 2x) \\
\partial_x \psi_i(x) &= \frac{2}{h_i^2}(2x - x_{i-1/2} - x_{i-1})
\end{aligned}$$

and the following integrals

$$\begin{aligned}
A_{11} &= \int_{x_{i-1}}^{x_i} (\partial_x \psi_{i-1})^2\,dx \\
A_{22} &= \int_{x_{i-1}}^{x_i} (\partial_x \psi_{i-1/2})^2\,dx \approx h_i(\partial_x \psi_{i-1/2}(x_{i-1/2}))^2 = 0 \\
A_{33} &= \int_{x_{i-1}}^{x_i} (\partial_x \psi_i)^2\,dx \\
A_{12} = A_{21} &= \int_{x_{i-1}}^{x_i} \partial_x \psi_{i-1} \cdot \partial_x \psi_{i-1/2}\,dx \approx h_i\left[(\partial_x \psi_{i-1} \cdot \partial_x \psi_{i-1/2})(x_{i-1/2})\right] = 0 \\
A_{13} = A_{31} &= \int_{x_{i-1}}^{x_i} \partial_x \psi_{i-1} \cdot \partial_x \psi_i\,dx \\
A_{23} = A_{32} &= \int_{x_{i-1}}^{x_i} \partial_x \psi_{i-1/2} \cdot \partial_x \psi_i\,dx \approx h_i\left[(\partial_x \psi_{i-1/2} \cdot \partial_x \psi_i)(x_{i-1/2})\right] = 0
\end{aligned}$$

At last we write the results in our matrix,

$$A = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & 0 & 0 \\ A_{31} & 0 & A_{33} \end{pmatrix} \tag{26}$$

Example 2

The second example handles also with the Laplace equation but in 2D. It was implemented by the authors of DEAL.II. Our focuss is on two lines in this program:

```
// m' : degree of the finite elements
fe (m'),
```

and

```
// Gauss formula, where n gives the number of notes in each direction
QGauss<2> quadrature_formula(n)
```

The degree of Gauss quadrature is given by $r' = 2n - 1$. We made some calculations for different m' and r'. The results are

n	1	2	3	4
r′	1	3	5	7
m′				
1	OK	OK	OK	OK
2	F	OK	OK	OK
3	F	ok	OK	OK
4	F	F	ok	OK
5	F	F	ok	ok
6	F	F	F	ok
7	F	F	F	ok
8	F	F	F	F

We shortly explain the notation:

- x: not interesting
- ok: normal convergence is given
- OK: Optimal convergence is given with $r' \geq 2m - 1$
- F: no convergence

This table prooves the theoretical part of the report.

References

[1] Rolf Rannacher, Methode der finiten Elemente
Skript 1976/77

[2] Rolf Rannacher, Numerik Partieller Differentialgleichungen
Skript SS 2006

[3] Hans-Juergen Reinhardt, Die Methode der finiten Elemente
Skript WS 1995/96

[4] Dietrich Braess, Finite Elemente, 4.,ueberarbeitete und erweiterte Auflage
Springer-Verlag 2007

[5] Guenther Haemmerlin, Karl-Heinz Hoffmann, Numerische Mathematik
Springer-Verlag 1989

[6] Peter Knabner, Lutz Angermann, Numerik partieller Differentialgleichungen
Springer-Verlag 2000

[7] Konrad Koenigsberger, Analysis 2, 4.,ueberarbeitete Auflage
Springer-Verlag 2002